# الرياضيات

**Secondary Mathematics Year I & II**

**First Edition**

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SECONDARY MATHEMATICS

# Preface

## About This Book

A rigorous, concept-driven course from algebra to calculus | Years 1 & 2 | Secondary Mathematics

This textbook is designed to teach the core concepts and techniques of secondary mathematics — from algebra and functions to calculus, geometry, and statistics — as a coherent and rigorous subject. While computational skill is essential, equal emphasis is placed on understanding definitions, theorems, and problem-solving strategies, as well as on reading, constructing, and justifying logical arguments (including proofs where appropriate). A clear organisation, fully worked examples, detailed solutions to many exercises, and a graded system of problems — ranging from basic recall to challenging proofs — will gradually equip the student with the tools and confidence to study advanced mathematics in a disciplined way.

Most students using this book will be in their first or second years of secondary education in the scientific or experimental sciences streams. A typical student will have completed basic arithmetic and introductory algebra. No prior knowledge of calculus is required, as the later chapters introduce limits and derivatives from first principles. With complete worked examples, step-by-step explanations, and solutions to a large majority of the exercises, the book is suitable for classroom use, independent study, and revision for national examinations (such as the Algerian Baccalaureate).

The book covers ten broad areas: numbers and algebra, functions, limits, derivatives, sequences, trigonometry, analytic geometry (both plane and space), vectors, plane geometry with transformations, and statistics with probability. The approach is consistently logical and concept-driven, with an emphasis on understanding *why* a method works, not just *how* to apply it. Historical vignettes place each topic in context, showing the human story behind the mathematics. Opening puzzles and real-world applications demonstrate the relevance of the material. Common mistakes are explicitly highlighted to help students avoid frequent pitfalls.

## How to Use This Book

The book is divided into ten chapters, each focusing on a major branch of mathematics. Within each chapter, sections are organised around core topics. Each section contains definitions, important results (presented as theorems or key facts), worked examples, and explanatory commentary. If you are enrolled in a course, read the section before class and attempt the opening puzzle. Use the worked examples as models for solving the exercises.

| | |
|---|---|
| **Tier 1**<br>*Instant Recall* | Direct application of definitions and formulae. These mirror the worked examples exactly and build confidence. |
| **Tier 2**<br>*Medium* | Multi-step problems that combine techniques or apply them in less familiar contexts. Many include graphs to read or interpret. |
| **Tier 3**<br>*Difficult (Challenge)* | Problems that require sustained reasoning, proofs, parametric analysis, or connections between topics. Designed for students aiming for mastery and top examination scores. |

Larger problem numbers generally indicate greater difficulty, though the tier label is the primary guide. Many exercises have complete solutions available on the companion platform. **Do not peek at the solution early** — working the exercises diligently is the best way to master the material. On the companion platform, you must submit your own attempt before the solution is unlocked.

| | |
|---|---|
| **Exam-style questions** | Appear at the end of each chapter. Modelled on actual baccalaureate examinations and require full justification of each step. Practising these under timed conditions is excellent preparation for high-stakes assessments. |
| **Chapter summaries** | List key results for quick revision. A list of exercise sources and further reading is provided at the end of each chapter, acknowledging the textbooks and online resources from which problems have been adapted. |
| **Historical briefs** | Open each chapter, placing the mathematics in its cultural and intellectual context. These show how the ideas developed over centuries and why they matter — they are not merely decorative. |
| **Opening puzzles** | Designed to be accessible yet thought-provoking. They are resolved at the end of the relevant section, rewarding the attentive reader. |
| **Common mistakes** | Boxes warn against typical errors. Read them carefully — they capture the most frequent points of confusion observed in classrooms. |

## To the Instructor

This book is designed to be flexible. The three tiers of exercises allow you to assign different levels to different students or to use the tiers for scaffolding. The computational problems (Tier 1) ensure basic fluency; the medium problems (Tier 2) develop analytical skills; the challenge problems (Tier 3) prepare students for proof-based mathematics. The historical and puzzle elements can be used to spark curiosity.

The material aligns closely with the Algerian secondary curriculum for the scientific and experimental sciences streams, but it is also suitable for any pre-calculus or introductory calculus course. Chapters are largely independent, allowing you to reorder topics as needed.

## Acknowledgements

The problems and examples have been adapted from a wide range of sources, including French textbooks (Barbazo, Hecht & Perpère, Repères), Anglophone texts (Stewart, OpenStax), competition resources (Art of Problem Solving), online notes (Paul’s Online Math Notes, Corbettmaths), and Algerian baccalaureate past papers. We are grateful to the original authors for their clear exposition and clever problems. Any errors or omissions are entirely our own.

We welcome feedback from students and instructors. Mathematics is a living discipline, and this text will continue to evolve.

> *“The mathematician does not study mathematics because it is useful; he studies it because he delights in it, and he delights in it because it is beautiful.”*
>
> **— Henri Poincaré**

CHAPTER 1

# Numbers & Algebra

The Language and Structure of Mathematics

## A Brief History: From Counting Stones to the Real Line

Mathematics begins with counting — and counting begins with need. The earliest known mathematical records, clay tablets from Mesopotamia dating to around 3000 BCE, are not philosophical treatises. They are receipts, inventories, and ration calculations. The Sumerians needed to track grain, livestock, and labour. Numbers were tools before they were ideas.

Yet within a few centuries, something remarkable happened: mathematicians began to ask questions that no inventory required. The Babylonians, working around 1800 BCE, discovered that some lengths could not be expressed as ratios of whole numbers. If a square has a side of length $1$, its diagonal has length $\sqrt{2}$ — a number that, as the Greeks would later prove, cannot be written as $\frac{p}{q}$ for any integers $p$ and $q$. This was the first glimpse of the irrational numbers, and it deeply unsettled those who encountered it.

The ancient Greeks organised these discoveries into the first systematic treatment of number and magnitude. Euclid's Elements (c. 300 BCE) established rigorous proofs for properties of integers and ratios. Eudoxus of Cnidus developed a theory of proportion that could handle irrational quantities without directly naming them — a workaround that stood for two thousand years.

The systematic study of algebra as a discipline separate from geometry began in the Islamic Golden Age. **Muhammad ibn Musa al-Khwarizmi** (c. 780–850 CE), working in the House of Wisdom in Baghdad, wrote *Al-Kitab al-Mukhtasar fi Hisab al-Jabr wal-Muqabala* (The Compendious Book on Calculation by Completion and Balancing). This work gave the world the word **algebra** (from *al-jabr*) and presented the first systematic solutions of linear and quadratic equations. Al-Khwarizmi also introduced Hindu-Arabic numerals to the Islamic world and, through later translations, to Europe. His methods for solving equations using geometric diagrams foreshadowed the later marriage of algebra and geometry.

Another scholar, **Al-Kindi** (c. 801–873 CE), wrote extensively on arithmetic and the use of Indian numerals, helping to establish the decimal place-value system.

The concept of **negative numbers** came much later, and from the East. Indian mathematicians, particularly Brahmagupta in the 7th century CE, developed rules for computing with negative quantities, treating them as debts in contrast to assets. European mathematicians resisted negatives well into the 17th century, calling them "absurd" or "fictitious" numbers. Rene Descartes, in 1637, still called negative roots of equations fausses — false.

Zero has its own remarkable history. The Babylonians had a placeholder symbol for an empty position but no concept of zero as a number. It was again Indian mathematicians — Brahmagupta chief among them — who first treated zero as a genuine number with its own arithmetic rules, around 628 CE. His rule that $\boldsymbol{a - a = 0}$ seems trivial today; at the time, it was a conceptual revolution.

The real number line as we now understand it — a complete, ordered continuum containing all rationals and irrationals — was not fully formalised until the 19th century. Richard Dedekind (1872) and Georg Cantor independently constructed the reals from the rationals using what are now called Dedekind cuts and Cauchy sequences respectively. Dedekind's famous introduction to his construction begins: "Whenever I speak of arithmetic, I mean the science of numbers in all its generality." That generality — that desire to include every possible number — is what gives us the real line.

**Why it matters for this chapter:** Every equation we write, every inequality we solve, every expression we factorise — all of it takes place within the system of real numbers, on the real line, subject to the rules that centuries of mathematicians worked to establish. Understanding the structure of that system is not a formality. It is the foundation.

**OPENING PUZZLE**

### The Diagonal of the Square

A farmer wants to fence a perfectly square field. Each side measures exactly 1 kilometre. He needs to run a diagonal wire from one corner to the opposite corner for extra support.

**Question 1:** What is the length of the diagonal? Write your answer exactly, without a calculator.

**Question 2:** Can this length be expressed as a fraction p/q where p and q are whole numbers? Try to find such a fraction, or explain why you think none exists.

**Question 3:** A different field has area 2 km$^2$. What is the length of its side? Is this number rational or irrational?

**By the end of Section 1.1, you will be able to answer all three questions rigorously — and you will understand why the ancient Greeks found this so disturbing.**

**BY THE END OF THIS CHAPTER, YOU WILL BE ABLE TO:**

- **Identify** and classify real numbers: natural, integer, rational, and irrational
- **Read and write** intervals in correct notation and represent them on the number line
- **Compute** with absolute values and solve equations and inequalities involving |x|
- **Simplify, expand, and factorise** algebraic expressions using standard techniques

- **Apply** the three remarkable identities confidently and in reverse
- **Solve** first-degree equations and inequalities in one variable
- **Solve** second-degree equations using the discriminant method
- **Construct sign tables** for linear and quadratic expressions and solve inequalities from them

# Section 1.1 — Sets of Numbers and the Real Line

## 1.1.1 — The Number Sets

Mathematics organises numbers into nested sets, each extending the previous to solve problems the previous set could not handle. The full hierarchy is:

| Symbol | Name | Members | Why it was needed |
|---|---|---|---|
| $\mathbb{N}$ | **Natural numbers** | $0, 1, 2, 3, 4,$ ... | Counting objects and measuring whole quantities |
| $\mathbb{Z}$ | **Integers** | ..., $-2, -1, 0, 1, 2,$ ... | Representing debts, temperatures below zero, directions |
| $\mathbb{Q}$ | **Rational numbers** | $\frac{p}{q}$ where $p$,$q$ in $\mathbb{Z}$, $q$ not $0$ | Dividing quantities: $\frac{1}{2}$ a loaf, 3/4 of a litre |
| $\mathbb{I}$ | **Irrational numbers** | $\sqrt{3}$, $e$, $\sqrt{2}$ , $\pi$ ... | Lengths, circles, and limits that rationals cannot reach |
| $\mathbb{R}$ | **Real numbers** | All of the above together | A complete, unbroken number line with no gaps |

The containment is strict: $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$. Every natural number is an integer; every integer is rational; every rational is real. But not every real number is rational — and that gap, filled by the irrationals, is what makes the real line complete.

### Key test — is a number rational?

A number is **rational** if and only if its decimal expansion either **terminates** (e.g. $\frac{1}{4} = 0.25$) or **repeats periodically** (e.g. $\frac{1}{3} = 0.333 \ldots$ $\frac{1}{7} = 0.142857142857 \ldots$. If a decimal neither terminates nor repeats — like $\pi = \mathbf{3.14159265}$... or $\sqrt{2} = \mathbf{1.41421356}$... — then the number is **irrational**.

### Worked Example 1.1.1 — Classifying Numbers

**Classify each of the following as $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, or irrational $\mathbb{I}$. Justify your answer.**

**(a)** $\mathbf{-7}$**:** It is a negative whole number. So $-7$ is in $\mathbb{Z}$, and also in $\mathbb{Q}$ and $\mathbb{R}$. It is **not** in $\mathbb{N}$.

**(b)** $\mathbf{\frac{3}{5}}$ **:** It is a ratio of two integers with non-zero denominator: $\frac{3}{5}$ in $\mathbb{Q}$. In decimal: $0.6$ (terminates). Confirmed rational.

**(c)** $\mathbf{\sqrt{9} = 3}$**:** This simplifies to $3$, which is in $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$.

**(d)** $\mathbf{\sqrt{5}}$: Suppose $\sqrt{5} = \frac{p}{q}$ in lowest terms. Then $5 = \frac{p^2}{q^2}$, so $p^2 = 5q^2$. This means $5$ divides $p^2$, hence $5$ divides $p$ (*If a prime number divides a square, it divides the* *number itself)*. Write $p = 5k$. Then $25k^2 = 5q^2$, so $q^2 = 5k^2$ meaning $5$ divides $q$ too. But then $p$ and $q$ share factor $5$, contradicting lowest terms. So $\sqrt{5}$ is **irrational**.

**(e) 0.727272...:** Let $x = 0.727272...$ Then $100x = 72.7272... = 72 + x$, so $99x = 72$, $x = \frac{72}{99} = \frac{8}{11}$. This is **rational**.

## 1.1.2 — Intervals on the Real Line

An interval is a connected subset of the real line — a set of all real numbers between two boundary values, with rules about whether the boundaries are included or excluded.

| Notation | Meaning | Example | On the number line |
|---|---|---|---|
| $[\boldsymbol{a},\boldsymbol{b}]$ | All $x$ with $a \le x \le b$ | $[\mathbf{1},\mathbf{4}]$ | Closed: both ends included |
| $]\boldsymbol{a},\boldsymbol{b}[$ | All $x$ with $a < x < b$ | $]\mathbf{1},\mathbf{4}[$ | Open: both ends excluded |
| $[\boldsymbol{a},\boldsymbol{b}[$ | All $x$ with $a \le x < b$ | $[\mathbf{1},\mathbf{4}[$ | Half-open: left included |
| $]\boldsymbol{a},\boldsymbol{b}]$ | All $x$ with $a < x \le b$ | $]\mathbf{1},\mathbf{4}]$ | Half-open: right included |
| $[\boldsymbol{a},+\infty[$ | All $x$ with $x \ge a$ | $[\mathbf{2},+\infty[$ | Unbounded right |
| $]-\infty,\boldsymbol{b}]$ | All $x$ with $x \le b$ | $]-\infty,\mathbf{3}]$ | Unbounded left |
| $]-\infty,+\infty[$ | All real numbers | $\mathbb{R}$ | The entire real line |

**Important:** We never write $+\infty]$ or $[-\infty$ with a square bracket. Infinity is not a real number — it cannot be included in an interval. Always use $]$ or $[$ (open) next to $+\infty$ or $-\infty$.

## 1.1.3 — Absolute Value

The **absolute value** of a real number $x$, written $|\boldsymbol{x}|$, is its distance from zero on the number line. Distance is always non-negative, so |x| $\ge$ 0 for all $x$.

### Definition — Absolute Value

$$|\boldsymbol{x}| = \boldsymbol{x} \quad \texttt{if } \boldsymbol{x} \ge \mathbf{0}$$

$$|\boldsymbol{x}| = -\boldsymbol{x} \quad \texttt{if } \boldsymbol{x} < \mathbf{0}$$

**Key properties:**

- $|\boldsymbol{x}| \ge \mathbf{0}$ for all $x$, with equality iff $x = 0$
- $|\boldsymbol{xy}| = |\boldsymbol{x}||\boldsymbol{y}|$
- $|\boldsymbol{x} + \boldsymbol{y}| \le |\boldsymbol{x}| + |\boldsymbol{y}|$ (triangle inequality)
- $|\boldsymbol{x} - \boldsymbol{y}|$ is the **distance** between $x$ and $y$ on the number line

- $|x| = a \leftrightarrow x = a \ \texttt{or} \ x = -a$ (for $a \geq 0$)
- $|x| < a \leftrightarrow -a < x < a$ (for $a > 0$)
- $|x| > a \leftrightarrow x > a \ \texttt{or} \ x < -a$ (for $a > 0$)

### Worked Example 1.1.2 — Solving Absolute Value Equations and Inequalities

**(a) Solve $|2x - 3| = 5$**

Either $2x - 3 = 5$ giving $x = 4$, or $2x - 3 = -5$ giving $x = -1$. Solution: $\boldsymbol{x = 4 \ \texttt{or} \ x = -1}$

**(b) Solve $|x + 1| < 4$**

This means $-4 < x + 1 < 4$, so $-5 < x < 3$. Solution: $\boldsymbol{x \in \ ]-5, 3[}$

**(c) Solve $|3x - 2| \geq 7$**

Either $3x - 2 \geq 7$ giving $x \geq 3$, or $3x - 2 \leq -7$ giving $x \leq -\frac{5}{3}$.

Solution: $\boldsymbol{x \in \ ]-\infty, -\frac{5}{3}] \cup [3, +\infty[}$

**(d) Solve $|x - 2| = |x + 5|$**

Squaring both sides:

$$\begin{aligned} (x-2)^2 &= (x+5)^2 \\ x^2 - 4x + 4 &= x^2 + 10x + 25 \\ -14x &= 21 \\ x &= -\frac{3}{2}. \end{aligned}$$

Solution: $\boldsymbol{x = -\frac{3}{2}}$.

### PUZZLE RESOLVED — The Diagonal of the Square

**Question 1:** By Pythagoras' theorem, the diagonal has length $\sqrt{\boldsymbol{1^2 + 1^2}} = \sqrt{\boldsymbol{2}}$.

**Question 2:** $\sqrt{2}$ is irrational. The proof: suppose $\sqrt{2} = \frac{p}{q}$ in lowest terms. Then $2 = \frac{p^2}{q^2}$, so $p^2 = 2q^2$. Thus $p$ is even — write $p = 2k$. Then $4k^2 = 2q^2$, so $q^2 = 2k^2$, meaning $q$ is also even. But then $p$ and $q$ share the factor 2, contradicting our assumption. So no such fraction $\frac{p}{q}$ exists.

**Question 3:** The field has area $2 = side^2$, so the $side = \sqrt{2}\,km$ — also irrational.

The ancient Greeks, who believed all quantities should be expressible as ratios of whole numbers, found this result so disturbing that, according to legend, the Pythagorean who first proved it was thrown into the sea.

## Section 1.2 — Algebraic Expressions

### 1.2.1 — Simplification and Expansion

An **algebraic expression** is a combination of numbers, variables, and operations. Simplifying an expression means writing it in a form that is equivalent but easier to work with — usually by expanding brackets and collecting like terms.

**Rules for expanding brackets:**

- $a(b + c) = ab + ac$ (distributivity)
- $(a + b)(c + d) = ac + ad + bc + bd$ (FOIL method)
- Collect like terms: terms with the same variable and exponent can be added or subtracted
- Simplify fractions by factoring numerator and denominator, then cancelling common factors

**Worked Example 1.2.1 — Expanding and Simplifying**

**(a) Expand and simplify:** $(3x - 2)(2x + 5) - 4x^2$

$$= 6x^2 + 15x - 4x - 10 - 4x^2 = 2x^2 + 11x - 10$$

**(b) Simplify:** $\frac{(x^2 - 4)}{(x^2 - x - 2)}$

Factor numerator: $(x - 2)(x + 2)$. Factor denominator: $(x - 2)(x + 1)$.

$= \frac{x+2}{x+1}$ for $x \neq 2$.

**(c) Simplify:** $(\frac{2}{x}) - (\frac{3}{x+1})$

Common denominator is $x(x + 1)$:

$$\frac{2(x+1)}{x(x+1)} - \frac{3x}{x(x+1)}$$

$$= \frac{2x + 2 - 3x}{x(x+1)}$$

$$= \frac{2 - x}{x(x+1)}$$

### 1.2.2 — Factorisation

Factorisation is the reverse of expansion: writing an expression as a **product** of simpler factors. It is one of the most important skills in algebra, used to solve equations, simplify fractions, and analyse functions.

**Standard factorisation techniques:**

- **Common factor:** $ab + ac = a(b + c)$ — always try this first
- **Grouping:** $ax + ay + bx + by = a(x + y) + b(x + y) = (a + b)(x + y)$
- **Difference of two squares:** $a^2 - b^2 = (a - b)(a + b)$
- **Quadratic trinomials:** $x^2 + (p + q)x + pq = (x + p)(x + q)$
- **Completing the square:** $ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + (c - \frac{b^2}{4a})$

### 1.2.3 — The Three Remarkable Identities

Three algebraic identities appear so frequently that they must be memorised and recognised instantly — both in the expanded form and the factored form.

**The three remarkable identities:**

**Identity 1 — Perfect square (sum):** $(a + b)^2 = a^2 + 2ab + b^2$

**Identity 2 — Perfect square (difference):** $(a - b)^2 = a^2 - 2ab + b^2$

**Identity 3 — Difference of squares**: $(a + b)(a - b) = a^2 - b^2$

These must be usable in both directions: to expand a product, and to factorise an expression that matches the pattern.

**Worked Example 1.2.2 — Remarkable Identities**

**(a) Expand** $(3x - 4)^2$

Using Identity 2 with $a = 3x, \quad b = 4$:

$$(3x)^2 - 2(3x)(4) + 4^2 = 9x^2 - 24x + 16$$

**(b) Factorise** $25x^2 - 49y^2$

Recognise as $(5x)^2 - (7y)^2$ — difference of squares:

$= (5x - 7y)(5x + 7y)$

**(c) Factorise** $4x^2 + 12x + 9$

Check: $(2x)^2 + 2(2x)(3) + 3^2 = (2x + 3)^2$. Confirmed perfect square.

**(d) Expand and simplify** $(x + 2)^2 - (x - 2)^2$

Use identities 1 and 2: $(x^2+4x+4)-(x^2-4x+4) = 8x$. (Or: difference of squares on the original: $((x+2)-(x-2))((x+2)+(x-2)) = (4)(2x) = 8x.$)

**COMMON MISTAKES — Section 1.2**

| **Incorrect** | **Correct** |
|---|---|
| $(a + b)^2 = a^2 + b^2$ | $(a+b)^2 = a^2 + 2ab + b^2$. The middle term $2ab$ is always missing in this error. Try $a = 3,\ b = 4$: $(7)^2 = 49$ but $9 + 16 = 25$. |
| $\frac{x^2-4}{x-2} = x^2 - 2$ | You cannot cancel like this. Factor first: $\frac{(x-2)(x+2)}{x-2} = x + 2$. Only factors cancel, never terms. |

## Section 1.3 — First-Degree Equations and Inequalities

### 1.3.1 — First-Degree Equations

A **first-degree equation** (or linear equation) in one variable $x$ is any equation that can be written in the form $\boldsymbol{ax = b}$ where $\boldsymbol{a \neq 0}$. The solution is unique: $\boldsymbol{x = \frac{b}{a}}$. The strategy is to isolate $x$ by performing identical operations on both sides.

**Worked Example 1.3.1 — Solving Linear Equations**

**(a) Solve** $3(2x - 1) - 4 = 2(x + 3)$

$$\boldsymbol{6x - 3 - 4 = 2x + 6 \quad \rightarrow \quad 4x = 13 \quad \rightarrow \quad x = \frac{13}{4}}$$

**(b) Solve** $\boldsymbol{\frac{x+2}{3} - \frac{x-1}{4} = 1}$

Multiply through by 12: $4(x + 2) - 3(x - 1) = 12$

$$\boldsymbol{4x + 8 - 3x + 3 = 12 \quad \rightarrow \quad x + 11 = 12 \quad \rightarrow \quad x = 1}$$

**(c) Solve** $\boldsymbol{|2x - 1| = 5x - 3}$ **(note: absolute value equation)**

Case 1: $2x - 1 = 5x - 3 \rightarrow 2 = 3x \rightarrow x = \frac{2}{3}$. Check: $\left|\frac{4}{3} - 1\right| = \frac{1}{3}$; $5\left(\frac{2}{3}\right) - 3 = \frac{1}{3}$. Valid.

Case 2: $-(2x - 1) = 5x - 3 \rightarrow 1 - 2x = 5x - 3 \rightarrow 4 = 7x \rightarrow x = \frac{4}{7}$. Check: $|\frac{8}{7} - 1| = \frac{1}{7}$; $5\left(\frac{4}{7}\right) - 3 = \frac{20}{7} - 3 = -\frac{1}{7}$. Since $5x - 3$ must be $\geq 0$ for this to equal |...|, and $-\frac{1}{7} < 0$: **rejected**.

Solution: $\boldsymbol{x = \frac{2}{3}}$ `only.`

### 1.3.2 — First-Degree Inequalities

The rules for solving inequalities are the same as for equations, with one critical exception: **multiplying or dividing both sides by a negative number reverses the direction of the inequality sign**. This is the most common source of errors in this topic.

**Rules for inequalities:**

- Adding or subtracting any number from both sides: inequality direction unchanged
- Multiplying or dividing by a **positive** number: direction unchanged
- Multiplying or dividing by a **negative** number: **direction reverses** ( $<$ becomes $>$ , $\leq$ becomes $\geq$ )
- The solution is typically an interval or union of intervals

### Worked Example 1.3.2 — Solving Linear Inequalities

**(a) Solve $\mathbf{3 - 2x < 7}$**

$\mathbf{-2x < 4}$

Divide by $-2$ : $\mathbf{x > -2}$. Solution: $]\mathbf{-2, +\infty}[$

**(b) Solve $\mathbf{-1 \leq \frac{2x+3}{5} < 3}$**

Multiply all parts by $5$: $-5 \leq 2x + 3 < 15$

Subtract $3$: $-8 \leq 2x < 12$

Divide by $2$: $\mathbf{-4 \leq x < 6}$. Solution: $[\mathbf{-4, 6}[$

**(c) Solve $\mathbf{|3x + 1| <= 5}$**

Equivalent to $-5 \leq 3x + 1 \leq 5$, so $-6 \leq 3x \leq 4$, giving $\mathbf{-2 \leq x \leq \frac{4}{3}}$. Solution: $[\mathbf{-2, \frac{4}{3}}]$

## 1.3.3 — Sign Tables for Linear Expressions

A **sign table** shows the sign (positive or negative) of an expression across its domain, organised by where the expression equals zero. For a linear expression $\boldsymbol{ax + b}$, the expression equals zero at $\boldsymbol{x = -\frac{b}{a}}$, is positive on one side and negative on the other.

### Worked Example 1.3.3 — Sign Table for a Linear Expression

**Build the sign table of $\boldsymbol{f(x) = 2x - 6}$ and solve $\mathbf{2x - 6 \geq 0}$.**

Root: $2x - 6 = 0 \to x = 3$.

Since the coefficient of x is positive (2 > 0), the expression is negative left of the root and positive right of it.

Solution to $2x - 6 \geq 0$: $\boldsymbol{x\ in}\ [\mathbf{3, +\infty}[$

# Section 1.4 — Second-Degree Equations and Inequalities

## 1.4.1 — The Discriminant Method

A **second-degree equation** (quadratic) has the form $ax^2 + bx + c = 0$ where $a \neq 0$. The key quantity is the **discriminant**, defined as $\Delta = b^2 - 4ac$. The sign of Delta determines the nature and number of solutions.

**The discriminant and its three cases:**

- $\Delta > 0$**:** Two distinct real roots: $x_1 = \frac{-b - \sqrt{\Delta}}{2a}$ and $x_2 = \frac{-b + \sqrt{\Delta}}{2a}$
- $\Delta = 0$: One repeated real root (double root): $x_0 = -\frac{b}{2a}$
- $\Delta < 0$**:** No real roots. The quadratic does not cross the x-axis.

When two roots exist, their sum and product satisfy Vieta's formulas:

- $x_1 + x_2 = -\frac{b}{a}$
- $x_1 \times x_2 = \frac{c}{a}$

### Worked Example 1.4.1 — Solving Quadratic Equations

**(a) Solve $2x^2 - 5x - 3 = 0$**

$$a = 2, \qquad b = -5, \qquad c = -3. \ \Delta = 25 + 24 = 49 > 0.$$

$$x_1 = \frac{5 - 7}{4} = -\frac{1}{2} \qquad x_2 = \frac{5 + 7}{4} = 3$$

Check with Vieta: $x_1 + x_2 = \frac{5}{2} = -\frac{b}{a}$ and $x_1 \times x_2 = -\frac{3}{2} = \frac{c}{a}$. Confirmed.

**(b) Solve $x^2 - 6x + 9 = 0$**

$\Delta = 36 - 36 = 0$. Double root: $x_0 = \frac{6}{2} = 3$. So $(x - 3)^2 = 0$.

**(c) Solve $x^2 + x + 1 = 0$**

$\Delta = 1 - 4 = -3 < 0$. No real solutions.

## 1.4.2 — Factorisation Using the Discriminant

When a quadratic has real roots $x1$ and $x2$, it can always be written in factored form:

$$ax^2 + bx + c = a(x - x_1)(x - x_2)$$

When there is a double root $x0$: $ax^2 + bx + c = a(x - x_0)^2$.

When $\Delta < 0$, the quadratic cannot be factored over the reals.

### 1.4.3 — Sign Tables for Quadratic Expressions

The sign of $ax^2 + bx + c$ depends on both the discriminant and the leading coefficient $a$. The sign table has a standard structure:

**Sign table rules for $ax^2 + bx + c$ :**

- $\Delta > 0, a > 0$: $(-)$ between $x_1$ and $x_2$, $(+)$ outside
- $\Delta > 0, a < 0$: $(+)$ between $x_1$ and $x_2$, $(-)$ outside
- $\Delta = 0, a > 0$: always $(+)$ except zero at $x_0$
- $\Delta = 0, a < 0$: always $(-)$ except zero at $x_0$
- $\Delta < 0, a > 0$: always strictly $(+)$
- $\Delta < 0, a < 0$: always strictly $(-)$

Memory aid: when $a > 0$ the parabola opens upward — it is positive outside and negative inside (between its roots). When $a < 0$ it opens downward — positive inside, negative outside.

**Worked Example 1.4.2 — Sign Table and Inequality**

**Solve the inequality: $-x^2 + 3x + 4 \geq 0$**

**Step 1 — Find roots:** $a = -1,\ b = 3,\ c = 4.\ \Delta = 9 + 16 = 25.$

$$x_1 = \frac{-3 - 5}{-2} = 4 \quad x_2 = \frac{-3 + 5}{-2} = -1$$

So $x_1 = -1\ and\ x_2 = 4$ (ordering them: $x_1 = -1 < x_2 = 4$).

**Step 2 — Sign table**: $a = -1 < 0$, so expression is positive BETWEEN the roots:

$$\begin{array}{lccccccc} x & -\infty & & -1 & & 4 & & +\infty \\ -x^2 + 3x + 4 & & (-) & 0 & (+) & 0 & (-) & \end{array}$$

**Step 3 — Solution:** We want $\geq 0$, which means the (+) zone plus the roots:

$$x\ in\ [-1, 4]$$

**COMMON MISTAKES — Sections 1.3 & 1.4**

| Incorrect | Correct |
|---|---|

| | |
|---|---|
| Divide both sides by $-3$: $-3x < 9 \rightarrow x < -3$ | Dividing by $-3$ reverses the sign: $-3x < 9 \rightarrow x > -3$. Always flip the inequality when dividing by a negative number. |

| **Incorrect** | **Correct** |
|---|---|
| $\Delta = b^2 - 4ac$ with $a = 2, b = 5, c = -3$: $\Delta = 25 - 4(2)(-3) = 25 - 24 = 1$ | $4ac = 4(2)(-3) = -24$, so $b^2 - 4ac = 25 - (-24) = 25 + 24 = 49$. Watch the sign of c carefully. |

| **Incorrect** | **Correct** |
|---|---|
| $a > 0$ and $\Delta > 0$, so $ax^2 + bx + c > 0$ everywhere except at the roots. | With $a > 0$ and $\Delta > 0$, the quadratic is NEGATIVE between the two roots and positive outside them. The parabola dips below zero between $x_1$ and $x_2$. |

# EXERCISES— CHAPTER 1

## Tier 1 — Direct Practice

These exercises mirror the worked examples exactly. Confidence at this level is required before moving to Tier 2.

### Number sets and intervals

1. **1.** Classify each number as an element of $\mathbb{N}\,, \mathbb{Z}\,, \mathbb{Q}\,,\ or \frac{\mathbb{R}}{\mathbb{Q}}$ (irrational). Choose the most restrictive set.

(a) $-5, 0.333\ldots, \sqrt{16}\,,\ \sqrt{7}\,, \frac{22}{7}, \pi\,, 0, -\frac{3}{4}$

**Source:** Barbazo, Mathématiques 2nde, Hachette (2014), Chapter 1, ex.3

1. **2.** Write each set in interval notation, then represent it on a number line sketch.

(a) All $x$ such that $-3 < x \leq 5$
(b) All $x$ such that $x \geq 7$
(c) All $x$ such that $x < -2\ or\ x > 4$
(d) All $x$ such that $|x| \leq 6$

**Source:** Paul's Online Math Notes, Algebra — tutorial.math.lamar.edu (2024)

1. **3.** Solve each absolute value equation or inequality. Write solutions in interval notation.

(a) $|x - 4| = 9$
(b) $|3x + 1| = 10$
(c) $|2x - 5| < 3$
(d) $|x + 2| \geq 5$
(e) $|4 - x| \leq 1$

**Source:** Corbettmaths, Absolute Value Worksheet — corbettmaths.com (2023)

### Algebra and identities

1. **4.** Expand and simplify fully.

(a) $(2x + 3)^2 - (2x - 3)^2$
(b) $(x + 1)(x - 1)(x^2 + 1)$
(c) $3(x - 2)^2 - (x + 1)(x - 4)$

**Source:** Barbazo, Mathématiques 2nde, Hachette (2014), p.28

1. **5.** Factorise completely.

(a) $x^2 - 25$
(b) $9x^2 - 12x + 4$
(c) $2x^2 + 7x + 3$
(d) $x^3 - x$

**Source:** Hecht & Perpere, Mathematiques Terminale S, Belin (2012), p.11

### Equations and inequalities

1. **6.** Solve each first-degree equation.
   (a) $5x - 3 = 2(x + 6)$
   (b) $\frac{x-1}{2} + \frac{x+3}{4} = 5$
   (c) $3(2 - x) - 4(x - 1) = 2x - 7$
   **Source:** OpenStax, College Algebra, Chapter 2.2 — openstax.org

## Tier 2 — Practice

These exercises require combining techniques or applying them in less familiar settings.

2. **7.** Prove that $\sqrt{3}$ is irrational, using the same method as for $\sqrt{2}$ in the chapter.
   **Source:** Classic result — see Stewart, Precalculus 7th ed., Brooks/Cole (2015), p.5

2. **8.** A recurring decimal $x = 0.123123123\ldots$ is claimed to be rational.
   (b) Let $x = 0.123123$... Multiply by $1000$ to find an equation for $x$, then solve for $x$ as a fraction.
   (c) Verify by dividing your fraction on a calculator.
   (d) Use this same method to convert $0.571428571428\ldots$ to a fraction.
   **Source:** Adapted from Art of Problem Solving, Pre-Algebra Chapter 4 — artofproblemsolving.com

2. **9.** Simplify the following expressions fully, stating any restrictions on the variable.
   (e) $\frac{x^2 - 9}{x^2 + x - 6}$
   (f) $\frac{2x^2 - 5x - 3}{2x^2 - 7x + 3}$
   (g) $\left(\frac{1}{x+1}\right) - \left(\frac{2}{x^2-1}\right)$
   **Source:** Barbazo, Mathematiques Terminale S, Hachette (2012), p.18, ex.22

2. **10.** Solve each quadratic equation. Compute the discriminant, state the nature of the roots, and verify using Vieta's formulas.
   (f) $3x^2 - 7x + 2 = 0$
   (g) $4x^2 - 12x + 9 = 0$
   (h) $x^2 - 2x + 5 = 0$
   (i) $(x - 3)(x + 2) = 14$
   **Source:** Reperes Terminale S, Nathan (2012), Chapter 1, ex.18

2. **11.** Solve each inequality using a sign table. Write solutions in interval notation.
   (d) $x^2 - 5x + 6 > 0$
   (e) $-2x^2 + 3x + 2 >= 0$
   (f) $(x - 1)(x + 3) < 0$
   (g) $x^2 + 4 > 0$
   **Source:** Barbazo, Mathematiques 1ere S, Hachette (2011), Chapter 2, ex.31

2. **12.** A quadratic $ax^2 + bx + c$ has roots $x_1 = -2$ and $x_2 = 5$, and passes through the point $(0, -20)$.
   - (e) Write the factored form of the quadratic in terms of $a$.
   - (f) Use the point $(0, -20)$ to find the value of $a$.
   - (g) Write the expanded form $ax^2 + bx + c$ and verify Vieta's formulas.
   - (h) Solve $ax^2 + bx + c \leq 0$ using a sign table.

   **Source:** Adapted from Hecht, Mathematiques Terminale S, Belin (2012), p.36

## Tier 3 — Challenge

These problems require sustained reasoning. There may be multiple methods. Show all working.

3. **13.** Let p and q be two real numbers such that $p + q = S$ and $pq = P$. Express the following in terms of $S$ and $P$ only:
   - (e) $\boldsymbol{p^2 + q^2}$
   - (f) $\boldsymbol{p^3 + q^3}$ -- Hint: use $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
   - (g) $\boldsymbol{(p - q)^2}$
   - (h) If $S = 5$ and $P = 3$, find $p^2 + q^2 and (p - q)^2$. Are $p$ and $q$ real? Justify.

   **Source:** Art of Problem Solving, Algebra Chapter 6 — artofproblemsolving.com/alcumus

3. **14.** Prove the following: for any real number $x$, $x^2 + x + 1 > 0$. (Hint: use the discriminant of the associated quadratic, or complete the square.)

   **Source:** Classic result — see Katz, A History of Mathematics, 3rd ed. (2009)

3. **15.** Find all real values of $k$ such that the equation $(k + 1)x^2 - 2kx + (k - 1) = 0$ has:
   - (h) Exactly one real solution
   - (i) Two distinct positive real solutions
   - (j) No real solutions

   **Source:** Adapted from Algerian Baccalaureate, Sciences Stream, 2017 — national past papers

3. **16. (Exploration)** The AM-GM inequality states: for positive reals $a$ and $b$, $\frac{a+b}{2}$ >= $\sqrt{ab}$.
   - (j) Prove this inequality algebraically.
   - (k) Give an example showing equality holds if and only if $a = b$.
   - (l) Use the inequality to show that for any positive $x$, $x + \frac{1}{x} \geq 2$.
   - (m) Verify part by checking $x = 1$, $x = 2$, $x = \frac{1}{3}$ numerically.

   **Source:** Classic inequality — see Stewart, Precalculus 7th ed., Brooks/Cole (2015), Appendix A

**EXAM-STYLE QUESTION**

Answer in full, showing all reasoning and justifying each step.

Consider the quadratic expression $\boldsymbol{f(x) = 2x^2 - (k + 3)x + k}$ where $k$ is a real parameter.

4. **1.** Compute the discriminant $\Delta$ of $f(x)$ as a function of $k$.
5. **2.** Find the values of $k$ for which $f$ has two distinct real roots.
6. **3.** Find the value of $k$ for which $f$ has a double root, and state that root.
7. **4.** For $k = 5$, solve the inequality $f(x) \leq 0$. Present your answer as an interval.
8. **5.** Show that for any value of $k$, the equation $f(x) = 0$ always has $x = 1$ as a solution.
9. **6.** Using part 5 and Vieta's formulas, find the second root as a function of $k$.

**Source:** Adapted from Algerian Baccalaureate, Sciences stream, 2019 and 2021 — national past papers

## CHAPTER SUMMARY — KEY RESULTS

### Section 1.1 — Numbers and the Real Line

- $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$. A number is rational iff its decimal expansion terminates or repeats.
- **Irrational numbers** have non-terminating, non-repeating decimals (e.g. $\sqrt{2}$, $\pi$).
- **Intervals:** $[a, b]$ closed; $]a, b[$ open; never write square brackets next to infinity.
- $\boldsymbol{|x| = a \leftrightarrow x = \pm a}$. $\boldsymbol{|x| < a \leftrightarrow -a < x < a}$. $\boldsymbol{|x| > a \leftrightarrow x > a}$ **or** $\boldsymbol{x < -a}$

### Section 1.2 — Algebraic Expressions

- **Three identities:** $\boldsymbol{(a+b)^2 = a^2 + 2ab + b^2}$; $\boldsymbol{(a-b)^2 = a^2 - 2ab + b^2}$; $\boldsymbol{(a+b)(a-b) = a^2 - b^2}$
- To cancel in a fraction, you must **factor** first — never cancel terms, only factors.

### Section 1.3 — First-Degree Equations and Inequalities

- **Multiplying/dividing by a negative number reverses the inequality sign.**
- **Sign table for $\boldsymbol{ax + b}$:** root at $x = -\frac{b}{a}$; expression positive on the same side as the leading coefficient.

### Section 1.4 — Second-Degree Equations and Inequalities

- $\Delta = \boldsymbol{b^2 - 4ac}$**:** $\Delta > 0$: two roots; $\Delta = 0$: double root; $\Delta < 0$: no real roots.
- **Vieta:** $\boldsymbol{x_1 + x_2 = -\frac{b}{a}}$ and $\boldsymbol{x_1 \times x_2 = \frac{c}{a}}$.
- **Sign of quadratic with $\boldsymbol{a > 0}$:** negative BETWEEN roots, positive OUTSIDE. Reversed if $a < 0$.
- **Factored form:** $\boldsymbol{ax^2 + bx + c = a(x - x_1)(x - x_2)}$ when $\Delta > 0$.

> "Algebra is the offer made by the devil to the mathematician. The devil says: I will give you this powerful machine and it will answer any question you like. All you need to do is give me your soul." — Michael Atiyah

### EXERCISE SOURCES & FURTHER READING

- **Barbazo, E.** Mathematiques 2nde & 1ere S. Hachette Education, 2011–2014.
- **Hecht & Perpere.** Mathematiques Terminale S. Belin, 2012.
- **Reperes Terminale S.** Nathan, 2012.
- **Stewart, J.** Precalculus: Mathematics for Calculus, 7th ed. Brooks/Cole, 2015.
- **OpenStax.** College Algebra. Open access: openstax.org, Chapter 2.
- **Art of Problem Solving.** Pre-Algebra and Algebra chapters: artofproblemsolving.com.
- **Paul's Online Math Notes.** Algebra section: tutorial.math.lamar.edu.
- **Corbettmaths.** Absolute value and algebra worksheets: corbettmaths.com.
- **Katz, V.J.** A History of Mathematics: An Introduction, 3rd ed. Addison-Wesley, 2009.
- **Algerian Baccalaureate past papers,** Sciences Stream, 2015–2022.

CHAPTER 1 — NUMBERS & ALGEBRA

# 50 Problems with Increasing Difficulty

Complete Exercise Bank

## Problems 1–10 — Instant Recall

Direct application of definitions and formulae. The method mirrors the worked examples exactly.

**Problem 1** Classify each number into the most specific set (`N, Z, Q,` or irrational). Write all sets it belongs to as well.

**(a)** $-8$
**(b)** $\sqrt{25}$
**(c)** $\frac{7}{3}$
**(d)** $0$
**(e)** $\sqrt{2}$

[Source: Barbazo, Mathematiques 2nde, Hachette (2014)]

**Problem 2** Write each set in interval notation, then list two numbers that belong to it and two that do not.

**(a)** All real $x$ such that $x \geq 3$
**(b)** All real $x$ such that $-2 < x \leq 7$
**(c)** All real $x$ such that $|\ x\ |< 5$

[Source: Paul's Online Math Notes, Algebra]

**Problem 3** Evaluate each absolute value expression without a calculator.

**(a)** $|3 - 8|$
**(b)** $|-7| + |4|$
**(c)** $|\pi - 4|$ (recall pi ≈ 3.14159...)
**(d)** $|-2| - |-9|$

[Source: Corbettmaths, Absolute Value]

**Problem 4** Expand and simplify using the three remarkable identities.

**(a)** $(x + 5)^2$
**(b)** $(3a - 2)^2$
**(c)** $(2x + 7)(2x - 7)$
**(d)** $(y - 4)^2$

[Source: Barbazo, Mathematiques 1ere S, Hachette (2011)]

**Problem 5** Factorise each expression completely. State which identity or technique you use.

**(a)** $x^2 - 36$

**(b)** $4t^2 - 12t + 9$

**(c)** $2x^2 + 6x$

**(d)** $49m^2 - 25n^2$

[Source: Hecht, Terminale S, Belin (2012)]

**Problem 6** Solve each first-degree equation. Check your answer by substitution.

**(a)** $3x - 7 = 11$

**(b)** $5 - 2x = -1$

**(c)** $4(x - 3) = 2x + 6$

**(d)** $\frac{x}{3} + 1 = 4$

[Source: OpenStax, College Algebra, Chapter 2]

**Problem 7** Solve each inequality and write the solution set as an interval.

**(a)** $2x - 5 > 3$

**(b)** $-3x + 1 \leq 10$

**(c)** $4 - x \geq 0$

**(d)** $2(x + 1) < 3x - 4$

[Source: Corbettmaths, Inequalities]

**Problem 8** Find the discriminant of each quadratic, then state the number of real roots (0, 1, or 2) without solving.

**(a)** $x^2 - 6x + 9$

**(b)** $2x^2 + 3x - 5$

**(c)** $x^2 + x + 3$

**(d)** $3x^2 - 12x + 12$

[Source: Barbazo, Terminale S, Hachette (2012)]

**Problem 9** Solve each quadratic equation using the discriminant formula.

**(a)** $x^2 - 7x + 12 = 0$

**(b)** $2x^2 + x - 3 = 0$

**(c)** $x^2 - 4x + 4 = 0$

**(d)** $3x^2 - 5x - 2 = 0$

[Source: Stewart, Precalculus 7th ed., Brooks/Cole (2015)]

**Problem 10** Build the sign table for each expression, then solve the corresponding inequality.

**(a)** $3x - 9 > 0$

**(b)** $(x - 2)(x + 5) \geq 0$

**(c)** $-2x + 4 < 0$

**(d)** $x^2 - 1 \leq 0$

[Source: Barbazo, 1ere S, Hachette (2011)]

## Problems 11–35 — Medium

Combine techniques, apply methods in less familiar settings, and reason through multi-step problems.

**Problem 11** A recurring decimal: let $x = 0.363636...$. Multiply by 100 to write an equation in x, then express x as a fraction in lowest terms. Verify on a calculator.

[Source: Art of Problem Solving, Pre-Algebra]

**Problem 12** Prove that $\sqrt{3}$ is irrational. Use the same method as for $\sqrt{2}$ : assume $\sqrt{3} = \frac{p}{q}$ in lowest terms and derive a contradiction.

[Source: Stewart, Precalculus 7th ed.]

**Problem 13** Solve each absolute value equation. Check for extraneous solutions.

(a) $|2x - 3| = |x + 5|$

(b) $|x^2 - 4| = 5$

(c) $|3x + 1| = 2x - 1$

[Source: Reperes Terminale S, Nathan (2012)]

**Problem 14** Expand and simplify fully. Show each step clearly.

(a) $(a + b + c)^2$

(b) $(x + 1)^2 - (x - 1)^2$

(c) $(2x - 3)^2 - (2x + 3)^2$

[Source: Barbazo, 2nde, Hachette (2014)]

**Problem 15** Factorise completely. Some require grouping or multiple steps.

(a) $x^3 - x$

(b) $2x^2 - 5x - 12$

(c) $x^4 - 16$

(d) $ax + ay - bx - by$

[Source: Hecht, Terminale S, Belin (2012)]

**Problem 16** Simplify each algebraic fraction. State restrictions on the variable.

(a) $\frac{(x^2 - 9)}{(x^2 - x - 6)}$

(b) $\frac{(2x^2 - 8)}{(x^2 + 2x - 8)}$

(c) $\frac{(x^2 - 5x + 6)}{(x^2 - 4)}$

[Source: Barbazo, Terminale S, Hachette (2012)]

**Problem 17** For each quadratic, complete the square, write it in the form $a(x - h)^2 + k$, and state the vertex and axis of symmetry.

(a) $x^2 - 6x + 2$

(b) $2x^2 + 8x - 3$

**(c)** $-x^2 + 4x + 5$

[Source: Stewart, Precalculus 7th ed.]

**Problem 18** Use Vieta's formulas — without solving — to answer the following about $3x^2 - 7x + 2 = 0$:

**(a)** Find the sum and product of the roots.

**(b)** Without computing the roots, find $x_1^2 + x_2^2$.

**(c)** Find $(x_1 - x_2)^2$ and hence determine $|x_1 - x_2|$.

[Source: Reperes Terminale S, Nathan (2012)]

**Problem 19** Solve the following equations. Pay careful attention to signs.

**(a)** $\frac{x+1}{2} - \frac{x-3}{4} = 5$

**(b)** $\frac{x}{x-1} = 2 + \frac{1}{x-1}$ (state the restriction first)

**(c)** $(2x-1)^2 = (x+3)^2$

[Source: Hecht, Terminale S, Belin (2012)]

**Problem 20** The temperature in a city one morning rose from `-3°C` to `12°C` over three hours. At noon it dropped back to `5°C`. Three inequalities are given about a temperature T:

**(a)** Which values of T satisfy $|T - 5| < 4$? Interpret in the context of temperature.

**(b)** Express the condition 'T is within 8 degrees of 0' as an absolute value inequality. Solve it.

**(c)** For which T is $|T + 3| > 10$?

[Source: OpenStax, College Algebra, Chapter 1]

**Problem 21** A quadratic $f(x) = x^2 - bx + c$ has two positive roots that sum to 6 and multiply to 8.

**(a)** Find b and c.

**(b)** Find the roots.

**(c)** Solve $f(x) \leq 0$.

[Source: Art of Problem Solving, Algebra]

**Problem 22** Solve the sign-table inequality: $\frac{(2x-6)(x+4)}{(x-1)} < 0$. Identify all critical values and exclude undefined points from your solution.

[Source: Barbazo, 1ere S, Hachette (2011)]

**Problem 23** Convert the following recurring decimals to fractions in lowest terms.

**(a)** $0.142857142857...$

**(b)** $1.272727...$

**(c)** $0.583333...$ (only the 3 repeats)

[Source: Art of Problem Solving, Pre-Algebra]

**Problem 24** Two athletes run the same 400m track. Athlete A completes a lap in $x$ seconds; Athlete B takes 5 seconds longer per lap.

**(a)** Write expressions for their speeds in m/s.

**(b)** If A runs at $8\ \mathrm{m/s}$, how many laps does B complete in 10 minutes while A completes exactly 12?

**(c)** For which values of $x$ is A's speed at least twice B's?

[Source: Adapted from Algerian Baccalaureate practice problems]

**Problem 25** Find all values of the parameter $k$ for which the equation $kx^2 + 4x + k = 0$ has:

**(a)** Exactly two distinct real roots.

**(b)** Exactly one real root.

**(c)** No real roots.

[Source: Algerian Baccalaureate, Sciences Stream, 2019]

**Problem 26** An integer $n$ is said to be perfect if it equals the sum of its proper divisors (excluding itself). Verify that 6 and 28 are perfect numbers. Then investigate: is $n = p \times q$ (product of two distinct primes) ever perfect? Justify.

[Source: Art of Problem Solving, Number Theory]

**Problem 27** Simplify each compound fraction.

**(a)** $\left(\frac{1}{x}+\frac{1}{y}\right)/\left(\frac{1}{x}-\frac{1}{y}\right)$

**(b)** $\frac{\left(x-\frac{1}{x}\right)}{\left(1+\frac{1}{x}\right)}$

**(c)** $\frac{\left(1+\frac{2}{x-1}\right)}{\left(1-\frac{2}{x+1}\right)}$

[Source: Stewart, Precalculus 7th ed.]

**Problem 28** Solve the double inequality, then verify your answer by testing a point inside and outside the solution set.

**(a)** $-3 < 2x - 1 \leq 5$

**(b)** $1 \leq \frac{3-x}{2} < 4$

**(c)** $|2x + 1| \leq 7$

[Source: Corbettmaths, Double Inequalities]

**Problem 29** A rectangular garden has perimeter 60 m. Let the width be $x$ metres.

**(a)** Express the area $A(x)$ as a quadratic in $x$.

**(b)** For which values of $x$ is the area at least 200 $m^2$ ?

**(c)** Complete the square to find the maximum area, and the dimensions that achieve it.

[Source: Adapted from Reperes Terminale S, Nathan (2012)]

**Problem 30** Without a calculator, determine which is greater, justifying each step algebraically.

**(a)** $\sqrt{5} + \sqrt{7}$ or $\sqrt{12}$

**(b)** $2\sqrt{3}$ or $\sqrt{11}$

[Source: Art of Problem Solving, Pre-Algebra]

**Problem 31** Expand and then factorise the result: $(a+b)^3$. Then use this to compute $1.02^3$ exactly without a calculator.

[Source: Stewart, Precalculus 7th ed.]

**Problem 32** A quadratic $f(x) = ax^2 + bx + c$ satisfies $f(0) = -3, f(1) = 2$, and $f(-1) = -6$. Find $a, b$, and $c$.

[Source: Hecht, Terminale S, Belin (2012)]

**Problem 33** Show that for any real number $x$, the expression $x^2 - x + 1$ is always strictly positive. Then determine the minimum value and where it is achieved.

[Source: Classic result — Barbazo, 1ere S]

**Problem 34** A cell phone plan charges a fixed monthly fee of 500 DA, plus 2 DA per minute for calls. A rival plan charges 200 DA/month plus 3 DA per minute.

**(a)** Let m = minutes used. Write cost functions for each plan.

**(b)** For which values of m is Plan A cheaper?

**(c)** If you use exactly 300 minutes per month, which plan do you choose and by how much?

[Source: Adapted from OpenStax, College Algebra]

**Problem 35** Show that the sum of any two consecutive odd numbers is divisible by 4. Then show that the sum of any four consecutive integers is divisible by 2 but not necessarily by 4.

[Source: Art of Problem Solving, Introduction to Algebra]

## Problems 36–50 — Difficult

Sustained reasoning, proofs, parametric problems, and connections between topics.

**Problem 36** Prove by contradiction that $\sqrt{2} + \sqrt{3}$ is irrational.

[Source: Art of Problem Solving, Number Theory]

**Problem 37** Let $p$ and $q$ be the roots of $x^2 - 5x + 3 = 0$. Without solving the equation, find:

**(a)** $p^2 + q^2$

**(b)** $p^3 + q^3$

**(c)** $\frac{1}{p} + \frac{1}{q}$

**(d)** $p^2 \times q + p \times q^2$

[Source: Reperes Terminale S, Nathan (2012)]

**Problem 38** Solve the system of inequalities simultaneously, writing the final solution as an intersection of intervals:

**System:** $2x^2 - 5x - 3 < 0$ AND $|x - 1| \leq 2$

[Source: Algerian Baccalaureate, Sciences Stream, 2018]

**Problem 39** Prove the AM-GM inequality: for any $a, b > 0, \frac{a+b}{2} \geq \sqrt{ab}$, with equality iff $a = b$.

**(a)** Prove it by showing $(a+b)^2 - 4ab = (a-b)^2 \geq 0$.

**(b)** Use part (a) to show that for any positive x, $x + \frac{1}{x} \geq 2$.

**(c)** Use part (a) to find the minimum value of $t + \frac{4}{t}$ for $t > 0$ and state where it occurs.

[Source: Classic inequality — Stewart, Precalculus 7th ed.]

**Problem 40** Find all integers $n$ such that $n^2 - 3n - 10$ is a perfect square. (A perfect square is a non-negative integer of the form $k^2$ for some non-negative integer k.)

[Source: Art of Problem Solving, Algebra]

**Problem 41** Let $f(x) = \frac{x^2+1}{x^2-1}$ for $Df = \mathbb{R} - \{-1; 1\}$.

**(a)** Show that $f(x) = 1 + \frac{2}{x^2-1}$.

**(b)** Study the sign of $x^2 - 1$ and deduce the sign of $\frac{2}{x^2-1}$.

**(c)** Hence determine all x for which $f(x) < 0$.

**(d)** Find all x for which $f(x) = 3$.

[Source: Algerian Baccalaureate, Sciences Stream, 2020]

**Problem 42** Prove that the equation $x^2 + y^2 = 3$ has no solution in rational numbers $x, y\ in\ \mathbb{Q}$.

[Source: Art of Problem Solving, Number Theory]

**Problem 43** A polynomial $P(x) = x^3 - ax^2 + bx - c$ has three positive roots $r, s, t$. Express $r^2 + s^2 + t^2$ and $\frac{1}{r} + \frac{1}{s} + \frac{1}{t}$ in terms of $a$, $b$, and $c$. Then find the range of $\mathbf{a}$ given that each root satisfies $1 < root < 4$.

[Source: Art of Problem Solving, Intermediate Algebra]

**Problem 44** Find all real values of $x$ satisfying the equation: $\sqrt{x+3} + \sqrt{6-x} = \sqrt{21}$

[Source: Art of Problem Solving, Algebra]

**Problem 45** Let $n$ be a positive integer. Prove that $n^2 + n$ is always even, and that $n^3 - n$ is always divisible by 6.

[Source: Art of Problem Solving, Introduction to Algebra]

**Problem 46** Solve the equation $x^4 - 5x^2 + 4 = 0$ by making the substitution $u = x^2$. Find all real solutions for x, then verify each one.

**Problem 47** For what values of $m$ does the equation $mx^2 - 2(m+1)x + (m+4) = 0$ have:

- **(a)** Two distinct real roots?
- **(b)** Two real roots of opposite sign?
- **(c)** Two positive real roots?

[Source: Algerian Baccalaureate, Sciences Stream, 2021]

**Problem 48** A number $N$ written in decimal is the concatenation of the first $k$ natural numbers: $N_k = 123456789101112....$ Determine $N_2 = 12, N_3 = 123$ modulo 9, and state whether each is divisible by 9. Then prove that $N_9 = 123456789$ is divisible by 9.

[Source: Art of Problem Solving, Number Theory]

**Problem 49** Prove the Cauchy-Schwarz inequality: for any reals $a, b, c, d$, $(ac + bd)^2 \leq (a^2 + b^2)(c^2 + d^2)$, with equality iff $ad = bc$. Then use it to show $\left(1 - \frac{2}{x+1}\right)$

[Source: Classic inequality — Katz, History of Mathematics]

**Problem 50** The **rational root theorem** states: if $P(x) = a_n \times x^n + \ldots + a_0$ has integer coefficients and $\frac{p}{q}$ (in lowest terms) is a root, then $p \mid a_0$ and $q \mid a_n$.

- **(a)** List all candidate rational roots of $P(x) = 6x^3 - 11x^2 - 26x + 15$.
- **(b)** Test candidates to find all rational roots.
- **(c)** Fully factorise $P(x)$.
- **(d)** Solve $P(x) > 0$.

[Source: Stewart, Precalculus 7th ed.]

CHAPTER 2

# Functions

Describing, Analysing, and Transforming
Mathematical Relationships

## A Brief History: The Idea of a Function

The word function is so familiar today that it is easy to forget how hard-won the concept was. For centuries, mathematicians worked with specific curves, specific formulas, and specific relationships — but without a general language to describe them all.

The story begins in the 17th century with René Descartes and Pierre de Fermat, who independently developed coordinate geometry — the idea of describing geometric curves using algebraic equations. A parabola was no longer just a shape; it was the equation $y = x^2$. A circle was $x^2 + y^2 = r^2$. Geometry and algebra, long treated as separate disciplines, were suddenly the same thing.

But it was Gottfried Wilhelm Leibniz who, around 1673, first used the word function to describe a quantity that changes along a curve. For Leibniz, a function was tied to geometry — it was a length, a slope, or a coordinate that varied as a point moved along a curve.

Leonhard Euler transformed the concept in the 18th century. His 1748 masterpiece Introductio in Analysin Infinitorum (Introduction to the Analysis of the Infinite) defined a function as an "analytic expression" — essentially, any formula built from numbers and operations. Euler also introduced the notation $\boldsymbol{f(x)}$, which we still use today. For Euler, functions were expressions first, curves second.

The modern definition — rigorous, geometric, and completely general — came from Johann Peter Gustav Lejeune Dirichlet in 1837. Dirichlet proposed that a function is simply any rule that assigns to each element of one set exactly one element of another set, with no requirement that the rule be expressible as a formula. This definition was so general that it unsettled many mathematicians of the time, but it is the one we use today.

**Why it matters:** The function is the central object of all modern mathematics. Every equation, every graph, every model in science, economics, and engineering is, at its heart, a function. Understanding what a function is — and how it behaves — is the foundation on which everything else in this book is built.

OPENING PUZZLE

### The Broken Vending Machine

A vending machine is programmed so that pressing button A always gives a cola, button B always gives water, and button C randomly gives either cola or water depending on how recently it was last used.

**Question 1:** Which buttons behave like functions? Which does not? Why?

**Question 2:** A student maps the buttons to drinks: A → cola, B → water, C → cola or water. Draw an arrow diagram for each. What property does the function diagram have that the non-function diagram lacks?

**Question 3:** The machine's revenue $R$ depends on the number of drinks sold $n$. If each drink costs 50 DA, write $R$ as a function of $n$. What is the domain of this function in the real world?

**We will return to and fully resolve these questions at the end of Section 2.1.**

**BY THE END OF THIS CHAPTER, YOU WILL BE ABLE TO:**

- **Define** a function precisely and identify whether a rule is a valid function
- **Find** the domain, image, and preimage of a function
- **Read** graphs and variation tables fluently
- **Recognise and use** the six reference functions and their transformations
- **Perform** operations on functions: sum, product, quotient, and composition
- **Identify** axes of symmetry and centres of symmetry of curves
- **Analyse** the variation of functions and find extrema from variation tables

## Section 2.1 — Foundations of Functions

### 2.1.1 — Definition and Notation

A **function** is a rule that assigns to every element of one set exactly one element of another set. This "exactly one" condition is what makes a function a function — and it is the condition that button C in the opening puzzle violates.

> **Definition — Function**
>
> A **function** $\boldsymbol{f}$ from a set $\boldsymbol{A}$ to a set $\boldsymbol{B}$ is a rule that assigns to each element $\boldsymbol{x}$ in $\boldsymbol{A}$ exactly one element $\boldsymbol{f(x)}$ in $\boldsymbol{B}$. We write:
>
> $$\boldsymbol{f : A \to B \ or \ x \to f(x)}$$
>
> - $\mathbf{A}$ is the **domain** of $\boldsymbol{f}$ — the set of all valid inputs
> - $\mathbf{B}$ is the **codomain** — the set of potential outputs
> - $\boldsymbol{f(x)}$ is the **image** of $\boldsymbol{x}$ under $\boldsymbol{f}$ — the output for input $\boldsymbol{x}$
> - The **range** (or image set) is the set of all actual outputs: $\{ f(x) : x \ in \ A \}$

In secondary mathematics, we almost always work with functions from $\mathbb{R}$ (or part of $\mathbb{R}$ ) to $\mathbb{R}$. The input is usually called $\boldsymbol{x}$ and the output $\boldsymbol{f(x), g(x), y}$, and so on.

### 2.1.2 — Domain of Definition

When a function is given by a formula, its **natural domain** is the largest subset of $\mathbf{R}$ on which the formula produces a real number. Two types of operations restrict the domain:

- **Division:** the denominator cannot be zero
- **Square roots (and even roots):** the expression under the root must be non-negative
- If a square root appears in the denominator, the radicand must be greater than zero.

> **Worked Example 2.1.1 — Finding the Domain**
>
> **Find the natural domain of:** **(a)** $\boldsymbol{f(x) = \frac{1}{x-3}}$ **(b)** $\boldsymbol{g(x) = \sqrt{2x+4}}$ **(c)** $\boldsymbol{h(x) = \frac{\sqrt{x+1}}{x-2}}$
>
> **(a)** The denominator $x-3=0$ when $x=3$. So: Domain $=$ $\mathbb{R}-\{3\} = ]-\infty, 3[\ U\ ]3,+\infty[$
>
> **(b)** We need $2x+4 \geq 0$, so $x \geq -2$. Domain = $[-2,+\infty[$
>
> **(c)** Two conditions: (i) $x + 1 \geq 0$ so $x \geq -1$; (ii) $x - 2 \neq 0$ so $x \neq 2$.
>
> $$Domain = [-1,+\infty[ - \{2\} = [-1,2[\ U\ ]2,+\infty[$$

### 2.1.3 — Image and Preimage

Given a function $\boldsymbol{f}$ and a value $\mathbf{a}$ in its domain, the **image** of $\boldsymbol{a}$ is the output $\boldsymbol{f}(\boldsymbol{a})$. Given a value $\boldsymbol{b}$ in the codomain, the **preimage** of $\boldsymbol{b}$ is the set of all inputs that map to $\boldsymbol{b}$: $\{\, x : f(x) = b \,\}$.

Finding a preimage means solving the equation $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{b}$ — a skill we will use extensively throughout this chapter.

**Worked Example 2.1.2 — Image and Preimage**

**Let $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{x}^{2} - \mathbf{3}$**. Find: (a) the image of 4; (b) the preimage of 1; (c) the preimage of -5.

**(a)** $\boldsymbol{f}(\mathbf{4}) = \mathbf{16} - \mathbf{3} = \mathbf{13}$. The image of 4 is 13.

**(b)** Solve $\boldsymbol{x}^{2} - \mathbf{3} = \mathbf{1}$, so $\boldsymbol{x}^{2} = \mathbf{4}$, giving $\boldsymbol{x} = \mathbf{2}\ \boldsymbol{or}\ \boldsymbol{x} = -\mathbf{2}$. Preimage of $1 = \{-2, 2\}$.

**(c)** Solve $\boldsymbol{x}^{2} - \mathbf{3} = -\mathbf{5}$, so $\boldsymbol{x}^{2} = -\mathbf{2}$. No real solution. Preimage of -5 = empty set.

## 2.1.4 — Reading Graphs

A function can be represented as a curve in the coordinate plane. Each point on the graph has the form $(x, f(x))$. Reading a graph means extracting information about the function from its visual representation.

**Key graph-reading skills:**

- **Domain:** the set of all $x - values$ for which the graph exists (horizontal extent)
- **Range:** the set of all $y - values$ the graph reaches (vertical extent)
- **Image of $\boldsymbol{a}$:** read the $y - coordinate$ at $x = a$
- **Preimage of $\boldsymbol{b}$:** find all x-values where the graph meets the horizontal line $y = b$
- **$\boldsymbol{x - intercepts}$:** where the graph crosses the $x - axis$ — these are solutions to $f(x) = 0$
- **$\boldsymbol{y - intercept}$:** where the graph crosses the $y - axis$ — this is $f(0)$, if $0$ is in the domain
- **Vertical line test:** a curve is the graph of a function if and only if every vertical line meets it at most once

## 2.1.5 — Variation Tables

A **variation table** summarises the behaviour of a function on its domain in a compact format. It shows: the key $x - values$ (where the function changes behaviour), the direction of change (increasing ↗ or decreasing ↘) on each interval, and the function's values at the boundaries.

**Worked Example 2.1.3 — Reading a Variation Table**

**The variation table of a function $\boldsymbol{f}$ is given below.**

$$\begin{array}{c|ccccccc}
\boldsymbol{x} & -\infty & & \mathbf{-2} & & \mathbf{1} & & +\infty \\
\boldsymbol{f(x)} & +\infty & \searrow & \mathbf{3} & \nearrow & \mathbf{8} & \searrow & -\infty
\end{array}$$

**(a)** On which intervals is $f$ increasing?

**(b)** What is the maximum value of $f$ on $[-2,1]$? Where is it reached?

**(c)** What is the minimum value of $f$ on $[-2,1]$?

**Solution: (a)** $f$ is increasing on $[-2,1]$ and decreasing on $]-\infty,-2]$ and $[1,+\infty[$.

**(b)** The maximum on $[-2,1]$ is $f(1) = 8$, reached at $x = 1$.

**(c)** The minimum on $[-2,1]$ is $f(-2) = 3$, reached at $x = -2$.

**PUZZLE RESOLVED — Part 1**

**Buttons A and B are functions**: each input (button press) produces exactly one output (drink). Button C is **not a function**: the same input can produce different outputs.

In an arrow diagram, a function has exactly one arrow leaving each input. Button C has two arrows from the same input — violating the definition.

For Part 3: $\boldsymbol{R(n) = 50n}$, where $n$ is the number of drinks sold. The natural domain of $R$ as a formula is all reals, but in context the domain is $\{0,1,2,3,\ldots\}$ — the non-negative integers.

## Section 2.2 — Reference Functions and Transformations

### 2.2.1 — The Six Reference Functions

Rather than analysing every new function from scratch, we build a library of **reference functions** whose behaviour is completely understood. Any function we encounter that can be related to one of these — through shifting, scaling, or reflection — inherits its known properties.

| Function | Domain | Variation | Key features |
|---|---|---|---|
| $f(x) = x$ | $]-\infty,+\infty[$ | Increasing on $\mathbb{R}$ | Passes through origin. Slope 1 everywhere. |
| $f(x) = x^2$ | $]-\infty+\infty[$ | Dec on $]-\infty,0]$, Inc on $[0,+\infty[$ | Minimum $f(0) = 0$. Symmetric about $y - axis$. |
| $f(x) = x^3$ | $]-\infty,+\infty[$ | Increasing on $\mathbb{R}$ | Inflection point at origin. Odd function. |
| $f(x) = \frac{1}{x}$ | $\mathbb{R} - \{0\}$ | Dec on $]-\infty,0[$ and on $]0,+\infty[$ | Never zero. Vertical asymptote $x = 0$; horizontal $y = 0$. |
| $f(x) = \sqrt{x}$ | $[0,+\infty[$ | Increasing on $[0,+\infty[$ | $f(0) = 0$. Grows slower and slower. |
| $f(x) = \lvert x\rvert$ | $]-\infty,+\infty[$ | Dec on $]-\infty,0]$, Inc on $[0,+\infty[$ | Minimum $f(0) = 0$. Sharp corner at origin. |

### 2.2.2 — Transformations of Functions

Given a reference function $f$, we can create new functions by applying transformations. Each transformation has a predictable, visual effect on the graph.

**Transformation rules — let $a, b, c, k$ be real constants:**

- $g(x) = f(x) + k$: vertical translation by $k$ units (upward if $k > 0$)
- $g(x) = f(x - a)$: horizontal translation by a units (rightward if $a > 0$)
- $g(x) = -f(x)$: reflection in the $x - axis$
- $g(x) = f(-x)$: reflection in the $y - axis$
- $g(x) = k \times f(x)$: vertical scaling by factor $k$ (stretches if $|k| > 1$, compresses if $|k| < 1$)
- $g(x) = f(bx)$: horizontal scaling by factor $1/b$
- Combinations: $g(x) = k \times f(b(x - a)) + c$ applies all transformations in order

#### Worked Example 2.2.1 — Describing Transformations

**Describe the transformations that map $f(x) = x^2$** onto each function below. State the vertex of each.

**(a)** $g(x) = (x-3)^2 + 1$: Translate $f$ right 3, up 1. Vertex: $(3, 1)$.

**(b)** $h(x) = -(x+2)^2$ : Translate f left 2, then reflect in $x - axis$. Vertex: (-2, 0), opening downward.

**(c) $\boldsymbol{p(x) = 2x^2 - 4x + 5}$**: Complete the square: $2(x-1)^2 + 3$. Translate right 1, up 3, stretch vertically by 2. Vertex: $(1, 3)$.

### 2.2.3 — Symmetry of Curves

A curve has an **axis of symmetry** if there is a vertical line about which the curve is a mirror image of itself. It has a **centre of symmetry** if there is a point about which rotating the curve 180° maps it onto itself.

**Key symmetry results:**

- **Even functions**: $f(-x) = f(x)$ for all $x$ in the domain. Graph symmetric about the $y - axis$. Examples: $x^2, |x|, cos(x)$.
- **Odd functions**: $f(-x) = -f(x)$ for all $x$ in the domain. Graph has the origin as a centre of symmetry. Examples: $x,\ x^3,\ sin(x), \frac{1}{x}$.
- A function can be neither even nor odd, which is the general case.
- For a translated function $\boldsymbol{g(x) = f(x-a) + b}$: the point $(a, b)$ is the centre of symmetry if $f$ is odd, or the axis $x = a$ is the axis of symmetry if f is even.

**Worked Example 2.2.2 — Symmetry**

**Determine whether each function is even, odd, or neither.**

**(a) $\boldsymbol{f(x) = 3x^4 - x^2 + 7}$**

$f(-x) = 3(-x)^4 - (-x)^2 + 7 = 3x^4 - x^2 + 7 = f(x)$. So $f$ is **even**. Axis of symmetry: $x = 0$.

**(b) $\boldsymbol{g(x) = 5x^3 - 2x}$**

$g(-x) = 5(-x)^3 - 2(-x) = -5x^3 + 2x = -(5x^3 - 2x) = -g(x)$. So $g$ is **odd**. Centre of symmetry: origin $(0, 0)$.

**(c) $\boldsymbol{h(x) = x^2 + x + 1}$**

$h(-x) = x^2 - x + 1$. This is neither $h(x)$ nor $-h(x)$. So $h$ is **neither**. (Its axis of symmetry is $x = -\frac{1}{2}$, found by completing the square.)

### 2.2.4 — Operations on Functions

Given two functions $\boldsymbol{f}$ and $\boldsymbol{g}$ with domains $\boldsymbol{D_f}$ and $\boldsymbol{D_g}$, we can define new functions on the common domain $\boldsymbol{D_f \cap D_g}$:

- $\boldsymbol{(f + g)(x) = f(x) + g(x)}$
- $\boldsymbol{(f - g)(x) = f(x) - g(x)}$

- $(\boldsymbol{f} \times \boldsymbol{g})(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x}) \times \boldsymbol{g}(\boldsymbol{x})$
- $\left(\frac{\boldsymbol{f}}{\boldsymbol{g}}\right)(\boldsymbol{x}) = \frac{\boldsymbol{f}(\boldsymbol{x})}{\boldsymbol{g}(\boldsymbol{x})}$, valid where $g(x) \neq 0$

## 2.2.5 — Composition of Functions

The **composition** of $\boldsymbol{g}$ after $\boldsymbol{f}$ is the function $(\boldsymbol{g} \circ \boldsymbol{f})(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{f}(\boldsymbol{x}))$. The output of $\boldsymbol{f}$ becomes the input of $\boldsymbol{g}$. The domain of $\boldsymbol{g} \circ \boldsymbol{f}$ is the set of $x$ in the domain of f for which $f(x)$ is in the domain of $g$.

### Worked Example 2.2.3 — Composition

**Let** $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{2x + 1}$ and $\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{x^2 - 3}$. Find $(g \circ f)(x)$ and $(f \circ g)(x)$.

$$\begin{aligned}(\boldsymbol{g} \circ \boldsymbol{f})(\boldsymbol{x}) &= g\big(f(x)\big) = g(2x+1) = (2x+1)^2 - 3 = 4x^2 + 4x + 1 - 3 \\ &= \boldsymbol{4x^2 + 4x - 2}\end{aligned}$$

$$(\boldsymbol{f} \circ \boldsymbol{g})(\boldsymbol{x}) = f\big(g(x)\big) = f(x^2 - 3) = 2(x^2 - 3) + 1 = \boldsymbol{2x^2 - 5}$$

Note: composition is not commutative in general — $(g \circ f) \neq (f \circ g)$.

**COMMON MISTAKES -- Sections 2.1 and 2.2**

| **Incorrect** | **Correct** |
|---|---|
| The domain of $\sqrt{x-3}$ is $x > 3$. | The domain is $x \geq 3$, written $[3, +\infty[. x = 3$ is included because $\sqrt{0} = 0$ is valid. |
| $(g\ o\ f)(x)$ means $g(x)$ times $f(x)$. | $(g\ o\ f)(x) = g(f(x))$ -- composition means substitution, not multiplication. Ex: Let $f(x) = x + 1, g(x) = x^2$. Then $(f \circ g)(2) = f(4) = 5$, but $(g \circ f)(2) = g(3) = 9$. Order matters!* |
| $f(-x) = -f(x)$ so $f$ is even. | $f(-x) = -f(x)$ means $f$ is ODD, not even. Even means $f(-x) = f(x)$. |

## Section 2.3 — Studying Variation

### 2.3.1 — Intervals of Increase and Decrease

We say a function $\boldsymbol{f}$ is **increasing** on an interval $\mathbb{I}$ if, for any $x_1 < x_2$ in $\mathbb{I}$, we have $f(x_1) < f(x_2)$. It is **decreasing** on $\mathbb{I}$ if $f(x_1) > f(x_2)$. Studying variation means identifying these intervals precisely.

**Formal definitions:**

- **Increasing on $\boldsymbol{I}$:** $\boldsymbol{x_1 < x_2 in\, \mathbb{I} \;=>\; f(x_1) < f(x_2)}$
- **Decreasing on $\boldsymbol{I}$:** $\boldsymbol{x_1 < x_2\ in\ \mathbb{I} \;=>\; f(x_1) > f(x_2)}$
- **Constant on $\boldsymbol{I}$:** $\boldsymbol{f(x_1) = f(x_2)}$ for all $x_1, x_2$ in $\mathbb{I}$
- **Local maximum at $\boldsymbol{c}$:** $\boldsymbol{f(c) \geq f(x)}$ for all $x$ near $c$
- **Local minimum at $\boldsymbol{c}$:** $\boldsymbol{f(c) \leq f(x)}$ for all $x$ near $c$

### 2.3.2 — Studying Variation Using Reference Functions

The most powerful technique for analysing variation without calculus is to rewrite a function in terms of a reference function, then use the known variation of that reference to deduce the variation of the new function.

The key tool is **completing the square** for quadratic functions, which rewrites $\boldsymbol{ax^2 + bx + c}$ in the form $\boldsymbol{a(x - h)^2 + k}$, making the vertex $(h, k)$ and the variation immediately visible.

**Worked Example 2.3.1 — Variation of a Quadratic (Completing the Square)**

**Find the variation of $\boldsymbol{f(x) = -2x^2 + 8x - 3}$ on $\mathbb{R}$.**

**Step 1 — Complete the square:**

$$\boldsymbol{f(x) = -2(x^2 - 4x) - 3 = -2(x^2 - 4x + 4 - 4) - 3 = -2(x-2)^2 + 8 - 3}$$
$$\boldsymbol{= -2(x-2)^2 + 5}$$

**Step 2 — Identify the reference:** The core is $\boldsymbol{(x-2)^2}$, a translation of $\boldsymbol{x^2}$, which has minimum at $x = 2$.

**Step 3 — Account for the coefficient -2:** The negative sign flips the variation. When $(x-2)^2$ increases, $f$ decreases, and vice versa.

**Step 4 — Write the variation table:**

| $\boldsymbol{x}$ | $-\infty$ | | $\boldsymbol{2}$ | | $+\infty$ |
|---|---|---|---|---|---|
| $\boldsymbol{f(x)}$ | $-\infty$ | ↗ | $\boldsymbol{5}$ | ↘ | $-\infty$ |

**Conclusion:** $f$ is increasing on $]-\infty, 2]$ and decreasing on $[2, +\infty[$. Maximum value is $f(2) = 5$.

### Worked Example 2.3.2 — Variation Involving 1/x

**Study the variation of $g(x) = \frac{2x+1}{x}$** on $]-\infty, 0[$ and $]0, +\infty[$.

**Step 1 — Rewrite:** $g(x) = \frac{2x}{x} + \frac{1}{x} = 2 + \frac{1}{x}$

**Step 2 — Reference:** This is a vertical translation of $\frac{1}{x}$ by $+2$. Since $\frac{1}{x}$ is decreasing on each piece of its domain, so is g.

**Step 3 — Variation table:**

$$\begin{array}{ccccccc} x & -\infty & & 0- & | & 0+ & & +\infty \\ g(x) & 2 & \searrow & -\infty & | & +\infty & \searrow & 2 \end{array}$$

Note: $g(x)$ approaches 2 from below as $x \to -\infty$, and from above as $x \to +\infty$. The line $y = 2$ is a horizontal asymptote.

### 2.3.3 — Maximum and Minimum Values on an Interval

To find the **maximum** and **minimum** values of a function on a closed interval $[a, b]$, we compare the values at:

- all turning points (local extrema) within $[a, b]$
- the endpoints: $f(a)$ and $f(b)$

The largest of these values is the maximum; the smallest is the minimum. Always check **both** endpoints and **all interior extrema** — students frequently forget one or the other.

**COMMON MISTAKES -- Section 2.3**

| **Incorrect** | **Correct** |
|---|---|
| $f$ is increasing so its values are positive. | Increasing describes direction of change, not sign of values. $f(x) = x - 100$ is increasing everywhere but $f(5) = -95$. |

| **Incorrect** | **Correct** |
|---|---|
| The max of $f$ on $[-3,4]$ is always at the vertex. | On a closed interval, always compare the vertex value with both endpoints. The maximum might be at an endpoint. |

**OPENING PUZZLE — Full Resolution**

**All three parts of the opening puzzle are now fully answerable:**

**Part 1 & 2** (resolved in Section 2.1): Buttons A and B are functions; button C is not. In an arrow diagram, a function has exactly one arrow from each input element.

**Part 3** (enriched by Sections 2.2–2.3): $R(n) = 50n$ is a linear function — a scaled version of the reference function $f(x) = x$. It is strictly increasing on its domain $\{0, 1, 2, \ldots\}$: the more drinks sold, the more revenue. There is no maximum (revenue is unbounded), but the minimum is $R(0) = 0$.

Extension: if the machine can hold at most 200 drinks, the domain becomes $\{0, 1, \ldots, 200\}$ and the maximum revenue is $R(200) = 10\ 000\ DA$.

# EXERCISES — CHAPTER 2

## Tier 1 — Direct Practice

These exercises mirror the worked examples directly. The method is the same; only the numbers change.

### Domain and notation

1. **1.** State the natural domain of each function:
2. **(a)** $f(x) = \frac{1}{x+5}$ **(b)** $g(x) = \sqrt{3x - 6}$ **(c)** $h(x) = \frac{1}{\sqrt{x-2}}$
3. **(d)** $p(x) = \frac{\sqrt{x+3}}{x-1}$ **(e)** $q(x) = \frac{x^2+1}{x^2-4}$

**Source:** Adapted from Terminale S exercises, Hecht & Perpère, Mathématiques, Belin (2012), p.34

1. **2.** Let $f(x) = x^2 - 5x + 6$. Calculate:
2. **(a)** $f(0)$ **(b)** $f(3)$ **(c)** $f(-1)$ **(d)** the preimage of 0 **(e)** the preimage of 2

**Source:** Adapted from Barbazo, Mathématiques 1ère S, Hachette (2011), Chapter 2

1. **3.** A variation table of function g is given below. Answer the questions.

$$x \quad -\infty \quad -3 \quad 0 \quad 2 \quad +\infty$$

$$g(x) \quad 1 \quad \nearrow \quad 5 \quad \searrow \quad -2 \quad \nearrow \quad 4 \quad \searrow \quad -\infty$$

2. **(a)** On which intervals is $g$ increasing?
3. **(b)** What is the maximum value of $g$ on $[-3, 2]$?
4. **(c)** What is $g(0)$? Is this a local minimum, local maximum, or neither?
5. **(d)** Does g have a global maximum on $]-\infty, +\infty[$? Justify.

**Source:** Paul's Online Math Notes, Functions Section — tutorial.math.lamar.edu (accessed 2024)

### Reference functions and transformations

1. **4.** For each, identify which reference function is the base, and describe the transformations applied:
2. **(a)** $f(x) = (x+4)^2 - 7$ **(b)** $g(x) = -\sqrt{x-1}$ **(c)** $h(x) = \frac{2}{x} + 3$
3. **(d)** $p(x) = |x+2| - 5$ **(e)** $q(x) = -(x-1)^3 + 4$

**Source:** Corbettmaths, Functions and Transformations Worksheets, corbettmaths.com (2023)

1. **5.** Determine whether each function is even, odd, or neither. Show all working.
2. **(a)** $f(x) = x^4 - 3x^2$ **(b)** $g(x) = x^3 + x$ **(c)** $h(x) = x^2 + x$ **(d)** $k(x) = |x^3|$

**Source:** Barbazo, Mathématiques Terminale S, Hachette (2012), Exercices p.48

## Tier 2 — Practice

These exercises require combining ideas from different sections of the chapter, or applying techniques in a less familiar form.

4. **6.** Let $f(x) = 2x^2 - 12x + 7$.
5. **(a)** Express $f(x)$ in the form $a(x - h)^2 + k$ by completing the square.
6. **(b)** State the vertex and the axis of symmetry of the graph of $f$.
7. **(c)** Construct the complete variation table of $f$ on $\mathbb{R}$.
8. **(d)** Find the minimum value of $f$. On which interval is it achieved?
9. **(e)** Solve $f(x) < 0$ graphically (indicate which part of the curve lies below the $x - axis$).

**Source:** Adapted from Repères Terminale S, Nathan (2012), p.62, ex.31

3. **7.** Let $f(x) = \sqrt{x}$ and $g(x) = x - 4$.
4. **(a)** State the domains of $f$ and $g$.
5. **(b)** Find $(f + g)(x)$ and state its domain.
6. **(c)** Find $(f \circ g)(x)$ and state its domain.
7. **(d)** Find $(g \circ f)(x)$ and state its domain.
8. **(e)** Are $(f \circ g)$ and $(g \circ f)$ equal? Justify.

**Source:** Art of Problem Solving, Precalculus Chapter 1 — artofproblemsolving.com

6. **8.** A function h is defined on $[-5, 3]$ with the following properties: $h$ is increasing on $[-5, -1]$, decreasing on $[-1, 2]$, increasing on $[2, 3]$; $h(-5) = 0, h(-1) = 6, h(2) = -2, h(3) = 4$.
7. **(a)** Sketch a possible variation table for $h$.
8. **(b)** Find the maximum and minimum values of $h$ on its domain.
9. **(c)** On which sub-interval is $h(x)$ guaranteed to be positive?
10. **(d)** Could $h$ be a quadratic function? Justify your answer.

**Source:** Adapted from Hecht & Perpère, Mathématiques Terminale S, Belin (2012), p.81

4. **9.** The function $f(x) = \frac{3x-2}{x+1}$ is defined on $\mathbb{R} - \{-1\}$.
5. **(a)** Rewrite $f(x)$ in the form $A + \frac{B}{x+1}$ where A, B are constants.
6. **(b)** Use this form to identify the reference function used.
7. **(c)** Deduce the variation table of $f$ on each piece of its domain.
8. **(d)** Identify the horizontal asymptote and the vertical asymptote of $f$.

**Source:** Barbazo, Mathématiques Terminale S, Hachette (2012), p.55

3. **10.** Let $f(x) = x^2 - 4x + 1$ and $g(x) = -x^2 + 2x + 5$.
4. **(a)** Find the $x - values$ where $f(x) = g(x)$.
5. **(b)** Determine the interval(s) where $f(x) < g(x)$.
6. **(c)** Find the maximum value of $g(x) - f(x)$ and state where it is reached.
7. **(d)** Interpret geometrically: what does the maximum of $g - f$ represent?

**Source:** Adapted from OpenStax Precalculus, Chapter 1.4, openstax.org (open access)

## Tier 3 — Challenge

These problems require genuine mathematical thinking. There may be no immediately obvious starting point. Read carefully, draw diagrams, and try small cases first.

10. **11.** Let $\boldsymbol{f(x) = ax^2 + bx + c}$ be a quadratic function with $a \neq 0$. Prove that f is always symmetric about the vertical line $\boldsymbol{x = -\frac{b}{2a}}$. (Hint: show that $\boldsymbol{f(-\frac{b}{2a} + t) = f(-\frac{b}{2a} - t)}$ for any real $t$.)

Source: Classic result — see Katz, A History of Mathematics, 3rd ed. (2009), Chapter 9

9. **12.** A function $f$ satisfies $\boldsymbol{f(x+y) = f(x) + f(y)}$ for all real $x, y$. Such a function is called additive.
10. **(a)** Show that $f(0) = 0$.
11. **(b)** Show that $f(-x) = -f(x)$ for all $x$. What does this mean about the symmetry of the graph?
12. **(c)** Show that $f(nx) = n \times f(x)$ for any positive integer $n$.
13. **(d)** If additionally $f$ is increasing on $\mathbb{R}$, prove that $f(x) = f(1) \times x$ for all rational $x$.

**Source:** Art of Problem Solving, Introduction to Functional Equations — artofproblemsolving.com/wiki

11. **13.** Let $\boldsymbol{f(x) = \frac{1}{1+x^2}}$.
12. **(a)** Show that $f$ is an even function and state its axis of symmetry.
13. **(b)** Show that $f(x)$ is always between 0 and 1 (exclusive at $0$, inclusive at $1$).
14. **(c)** Without using calculus, argue whether f is increasing or decreasing on $[0, +\infty[$. (Hint: compare $f(x_1)$ and $f(x_2)$ directly for $0 \leq x_1 < x_2$.)
15. **(d)** Sketch a careful variation table for $f$ on $\mathbb{R}$.

**Source:** Adapted from AOPS Precalculus Problem Set 2 — artofproblemsolving.com

9. **14. (Exploration)** The function $\boldsymbol{f(x) = x + \frac{1}{x}}$ is defined on $\mathbb{R} - \{\boldsymbol{0}\}$.
10. **(a)** Show that $f$ is odd.
11. **(b)** Show that for $x > 0, f(x) \geq 2$.
12. **(c)** Show that for $x < 0, f(x) \leq -2$.
13. **(d)** Construct the full variation table of $f$ on each part of its domain.
14. **(e)** What famous inequality does part (b) prove? (Research the AM-GM inequality.)

**Source:** Classic analysis problem — see Stewart, Precalculus: Mathematics for Calculus, 7th ed., Brooks/Cole (2015), p.233

**EXAM-STYLE QUESTION**

Answer in full, showing all reasoning. Write up as you would in a baccalaureate examination.

Let $\boldsymbol{f}$ be a function defined on $\mathbb{R}$ by $\boldsymbol{f(x) = x^2 - 6x + 5}$.

11. **1.** Express $f(x)$ in the form $\boldsymbol{(x - a)^2 + b}$, determining the constants $a$ and $b$.
12. **2.** Deduce the complete variation table of $f$ on $\mathbb{R}$, justifying each interval.
13. **3.** State the minimum value of $f$ and the value of $x$ at which it is achieved.
14. **4.** Determine the values of $x$ for which $f(x) \leq 0$.
15. **5.** Let $\boldsymbol{g(x) = f(x+3)}$. Describe the geometric transformation that maps the graph of $f$ to the graph of $g$. Write an explicit formula for $g(x)$ and state its vertex.

16. **6.** Determine whether $g$ is even, odd, or neither. Justify fully.

**Source:** Adapted from Baccalaureate Examination, Series Scientifique, Algeria 2019 — and Barbazo Terminale S, Hachette

CHAPTER SUMMARY — KEY RESULTS

### Section 2.1 — Foundations

- **Function:** a rule assigning exactly one output to each input. The vertical line test checks this graphically.
- **Natural domain:** exclude values causing division by zero or negative square roots.
- **Image and preimage:** $f(a)$ is the image of $a$; the preimage of $b$ is the solution set of $f(x) = b$.
- **Variation table:** shows increasing (↗), decreasing (↘), and boundary values in a single diagram.

### Section 2.2 — Reference Functions & Transformations

- **Six reference functions:** $\boldsymbol{x}, \boldsymbol{x^2}, \boldsymbol{x^3}, \boldsymbol{\frac{1}{x}}, \boldsymbol{\sqrt{x}}, \boldsymbol{|x|}$ — memorise their domains and variation.
- **Transformations:** $f(x-a)+k$ translates; $-f(x)$ reflects; $kf(x)$ scales. Apply in order: horizontal shift, scale, reflect, vertical shift.
- **Even (symmetric about y-axis):** $\boldsymbol{f(-x) = f(x)}$. Odd (centre of symmetry at origin): $\boldsymbol{f(-x) = -f(x)}$.
- **Composition:** $\boldsymbol{(g \circ f)(x) = g(f(x))}$. Not commutative in general.

### Section 2.3 — Studying Variation

- **Increasing on $\mathbb{I}$: $\boldsymbol{x_1 < x_2 \to f(x_1) < f(x_2)}$.** Decreasing: reverse the inequality.
- **Completing the square:** $\boldsymbol{ax^2 + bx + c = a(x-h)^2 + k}$, vertex at $(h, k)$, variation flips if $a < 0$.
- **Max/min on $[\boldsymbol{a}, \boldsymbol{b}]$:** compare all local extrema inside the interval AND both endpoints.
- **Rational functions:** rewrite as $A + \frac{B}{g(x)}$ to identify the reference and deduce variation.

"The mathematician does not study mathematics because it is useful; he studies it because he delights in it, and he delights in it because it is beautiful." — Henri Poincaré

EXERCISE SOURCES & FURTHER READING

- **Barbazo, E.** Mathématiques 1ère S & Terminale S. Hachette Éducation, 2011–2012.
- **Hecht & Perpère.** Mathématiques Terminale S. Belin, 2012.

- **Repères Terminale S.** Nathan, 2012.
- **Stewart, J.** Precalculus: Mathematics for Calculus, 7th ed. Brooks/Cole, 2015.
- **OpenStax.** Precalculus. Open access: openstax.org. Chapter 1 (Functions).
- **Art of Problem Solving.** Functional equations and precalculus topics: artofproblemsolving.com.
- **Paul's Online Math Notes.** Functions unit: tutorial.math.lamar.edu.
- **Corbettmaths.** Worksheet bank — Functions & Transformations: corbettmaths.com.
- **Katz, V.J.** A History of Mathematics: An Introduction, 3rd ed. Addison-Wesley, 2009.
- **Algerian Baccalaureate past papers,** Sciences Stream, 2016–2022. Available via the national education ministry portal.

CHAPTER 2 — FUNCTIONS

# 50 Problems with Increasing Difficulty

Complete Exercise Bank

## Problems 1–10 — Instant Recall

Direct application of definitions. No tricks — just recall the rule and apply it.

**Problem 1** State the natural domain of each function. Write your answer as an interval or union of intervals.

- **(a)** $f(x) = \frac{1}{x+4}$
- **(b)** $g(x) = \sqrt{3x - 9}$
- **(c)** $h(x) = \frac{1}{\sqrt{x-2}}$
- **(d)** $k(x) = \sqrt{\frac{(x+5)}{(x-1)}}$

[Barbazo, Mathematiques 1ere S, Hachette (2011)]

**Problem 2** Let $f(x) = 2x^2 - 3x + 1$. Compute each of the following.

- **(a)** $f(0)$
- **(b)** $f(2)$
- **(c)** $f(-1)$
- **(d)** The preimage(s) of 0 — i.e., solve $f(x) = 0$

[Paul's Online Math Notes, Functions — tutorial.math.lamar.edu]

**Problem 3** State whether each function is **increasing**, **decreasing**, or **neither** on the given interval, using the reference function table.

- **(a)** $f(x) = x^2$ on $]-\infty, 0]$
- **(b)** $g(x) = x^3$ on $\mathbb{R}$
- **(c)** $h(x) = \frac{1}{x}$ on $]0, +\infty[$
- **(d)** $k(x) = |x|$ on $]-\infty, 0]$

[Corbettmaths, Functions — corbettmaths.com]

**Problem 4** Describe in words the transformation that maps $f(x) = x^2$ to each function below. Then state the vertex of each.

- **(a)** $g(x) = (x - 3)^2$
- **(b)** $h(x) = x^2 + 5$
- **(c)** $p(x) = -x^2$

**(d)** $q(x) = (x + 2)^2 - 4$

[Stewart, Precalculus 7th ed., Brooks/Cole (2015)]

**Problem 5** Determine whether each function is **even**, **odd**, or **neither**. Show the test clearly.

**(a)** $f(x) = x^4 - 2x^2 + 1$

**(b)** $g(x) = x^3 - x$

**(c)** $h(x) = x^2 + x$

**(d)** $k(x) = |x^3|$

[Barbazo, Terminale S, Hachette (2012)]

**Problem 6** Let $f(x) = 3x - 1$ and $g(x) = x^2 + 2$. Compute each composed function and simplify.

**(a)** $(g \circ f)(x)$

**(b)** $(f \circ g)(x)$

**(c)** $(f \circ f)(x)$

**(d)** $(g \circ g)(2)$

[Art of Problem Solving, Precalculus]

**Problem 7** State the domain of each composed function. First find $(g \circ f)(x)$ explicitly, then determine the domain.

**(a)** $f(x) = \sqrt{x}, \quad g(x) = x - 3$

**(b)** $f(x) = x + 1, \quad g(x) = \frac{1}{x}$

**(c)** $f(x) = x^2, \quad g(x) = \sqrt{x}$

[Paul's Online Math Notes]

**Problem 8** A variation table for a function $h$ is given. Answer the questions below.

| $x$ | $-\infty$ | | $-2$ | | $3$ | | $+\infty$ |
|---|---|---|---|---|---|---|---|
| $h(x)$ | $-3$ | $\nearrow$ | $5$ | $\searrow$ | $-1$ | $\nearrow$ | $+\infty$ |

**(a)** On which intervals is h increasing?

**(b)** Find the maximum value of h on $[-2, 3]$.

**(c)** Find the minimum value of h on $]-\infty, +\infty[$. Does it exist?

**(d)** Is $h(-2) = 5$ a global maximum? Justify.

[Reperes Terminale S, Nathan (2012)]

**Problem 9** Apply the operations on functions. Let $f(x) = x + 1$ and $g(x) = x^2 - 1$.

**(a)** Write $(f + g)(x)$ and state its domain.

**(b)** Write $(\frac{f}{g})(x)$ and state its domain (exclude where denominator = 0).

**(c)** Simplify $(\frac{f}{g})(x)$ by factoring. What does this reveal?

**(d)** Compute $(f \cdot g)(3)$

[Barbazo, 1ere S, Hachette (2011)]

**Problem 10** Complete the variation table for $f(x) = -x^2 + 4x - 1$ by first completing the square.

Show that $f(x) = -(x-2)^2 + 3$, then fill in all entries of the variation table:

| | | | | | |
|---|---|---|---|---|---|
| $x$ | $-\infty$ | | $?$ | | $+\infty$ |
| $f(x)$ | $???$ | $\nearrow$ | $???$ | $\searrow$ | $???$ |

[Hecht, Terminale S, Belin (2012)]

## Problems 11–35 — Medium

Combine techniques. Several problems include graphs to read, analyse, or produce.

**Problem 11** Find the natural domain of each function. Justify every restriction.

**(a)** $f(x) = \sqrt{x^2 - 5x + 6}$

**(b)** $g(x) = ln\left(x^2 - 4\right)$ (ln is defined for positive arguments only)

**(c)** $h(x) = \frac{1}{\sqrt{x^2 - 9}}$

**(d)** $k(x) = \frac{\sqrt{4-x}}{\sqrt{(x+1)}}$

[Barbazo, Terminale S, Hachette (2012)]

**Problem 12** Let $f(x) = \frac{(x^2-4)}{(x-2).}$. Explain why $f$ is NOT defined at $x = 2$, then show that for $x \neq 2, f(x) = x + 2$. Sketch the graph of $f$ and clearly mark the **removable discontinuity**.

**(a)** Factor the numerator and simplify.

**(b)** What is $lim_{(x \to 2)} f(x)$? What is $f(2)$?

**(c)** Describe the graph: what does it look like, and what is missing?

[Stewart, Precalculus 7th ed.]

**Problem 13** Complete the square and fully analyse the variation of $f(x) = 2x^2 - 8x + 5$.

**(a)** Rewrite in vertex form $a(x-h)^2 + k$.

**(b)** State the vertex, axis of symmetry, and whether it is a minimum or maximum.

**(c)** Write the full variation table.

**(d)** Solve $f(x) \leq 0$

[Reperes Terminale S, Nathan (2012)]

**Problem 14** The graph below shows a function $f$. Answer all questions using the graph only — no formula is given.

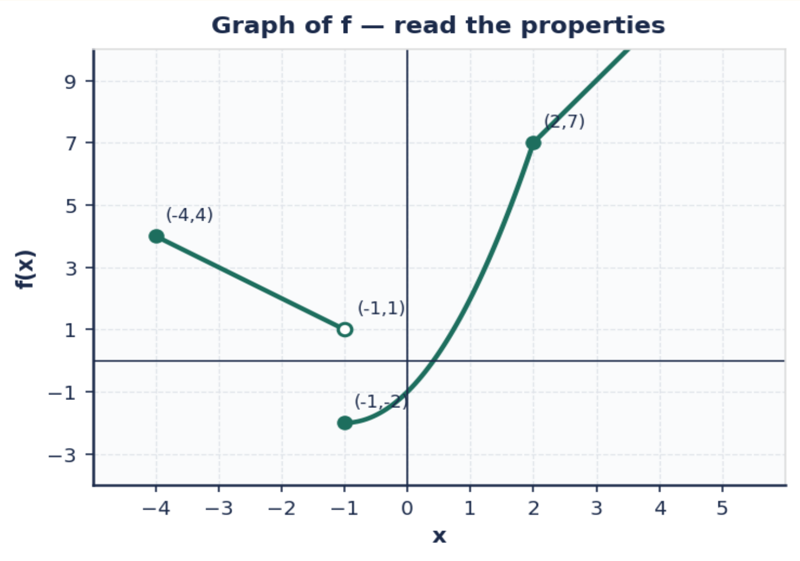


Figure: Graph of f — read properties from the curve

**(a)** State the domain and approximate range of $f$

**(b)** On which intervals is $f$ increasing? Decreasing?

**(c)** Identify any local maxima and minima. State their coordinates.

**(d)** Is $f(-1) = -2$ a local minimum? Is there a global minimum?

[Barbazo, Terminale S, Hachette (2012) — graph reading exercise]

**Problem 15** Determine the image set (range) of each function on the given domain.

**(a)** $f(x) = x^2 - 4$ on $[-2, 3]$

**(b)** $g(x) = \frac{1}{x}$ on $]0, 1]$

**(c)** $h(x) = \sqrt{x}$ on $[4, 9]$

[Paul's Online Math Notes]

**Problem 16** The curve below is the graph of a cubic function. Build its complete variation table.

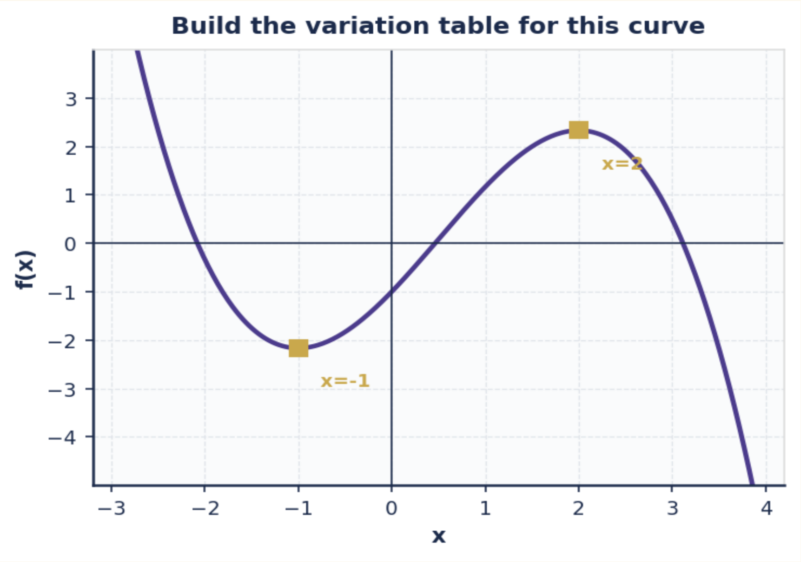


Figure: A cubic curve — build its variation table

The curve has critical points (stationary points) at $x = -1$ and $x = 2$.

- **(a)** Read off the local maximum and minimum values from the graph.
- **(b)** Complete the variation table with arrows ↗ and ↘.
- **(c)** Is this function even, odd, or neither? Justify from the graph.

[Hecht, Terminale S, Belin (2012) — variation table exercise]

**Problem 17** Let $f(x) = \frac{(2x-1)}{(x+3)}$. Rewrite $f$ in the form $A + \frac{B}{x+3}$ where A and B are constants.

- **(a)** Find A and B by performing the Euclidean division.
- **(b)** Identify the reference function and the transformation applied.
- **(c)** Deduce the variation table of $f$ on each part of its domain.
- **(d)** State the horizontal and vertical asymptotes.

[Barbazo, Terminale S, Hachette (2012)]

**Problem 18** Study the symmetry of the two functions shown in the figure.

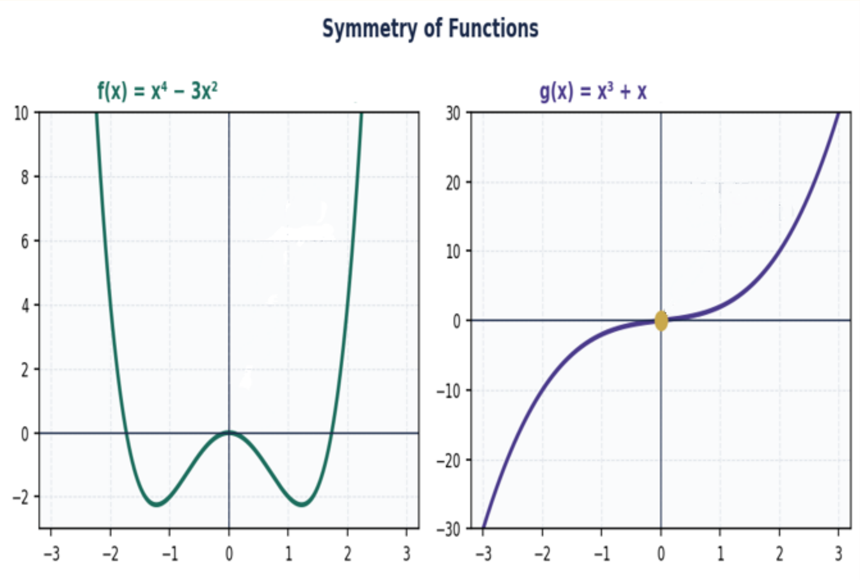


Figure: Even and odd function examples

**(a)** Verify algebraically that $f(x) = x^4 - 3x^2$ is even by computing $f(-x)$

**(b)** Verify that $g(x) = x^3 + x$ is odd by computing $g(-x)$

**(c)** What are the zeros of $f$? Of $g$? What geometric property do the zeros of an odd function always have?

**(d)** Write down a function that is neither even nor odd. Verify your claim.

[Barbazo, 1ere S, Hachette (2011)]

**Problem 19** Solve the following inequalities using variation tables or direct analysis. Write your answers as intervals.

**(a)** $x^2 - 2x - 8 > 0$

**(b)** $-3x^2 + 6x - 3 \geq 0$

**(c)** $\frac{x^2 - 1}{x - 3} < 0$

**(d)** $x^4 - 5x^2 + 4 \leq 0$

[Reperes Terminale S, Nathan (2012)]

**Problem 20** The graph below shows the function $g(x)$ (solid line) alongside $f(x) = x^2$ (dashed). Identify $g(x)$ completely.

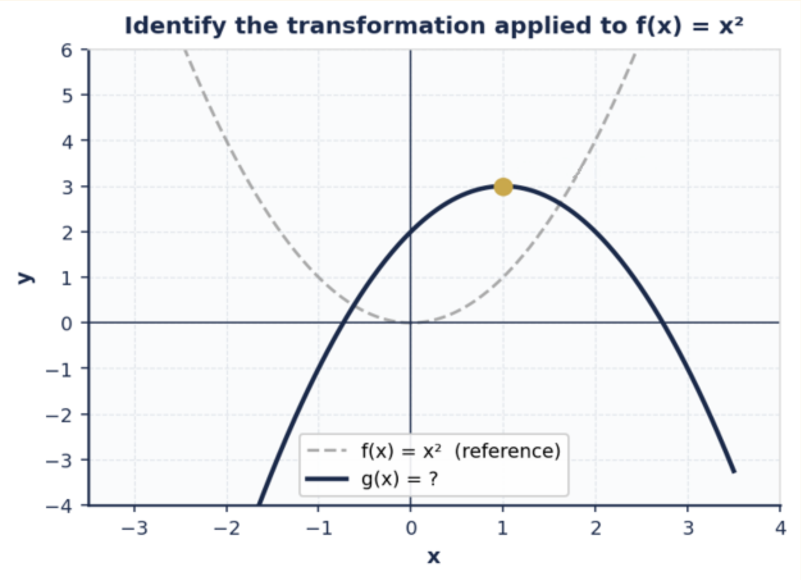


Figure: Identify the transformation — what is g(x)?

**(a)** Read off the vertex of $g$ from the graph.

**(b)** Determine whether $g$ opens upward or downward. What does this tell you about the leading coefficient?

**(c)** Write the equation of $g(x)$ in vertex form, then expand to standard form.

**(d)** Describe in words all the transformations applied to $f$ to obtain $g$

[Stewart, Precalculus 7th ed.]

**Problem 21** Let $f(x) = \sqrt{x + 2}$ and $g(x) = x^2 - 2$.

**(a)** Find $(f \circ g)(x)$ and its domain.

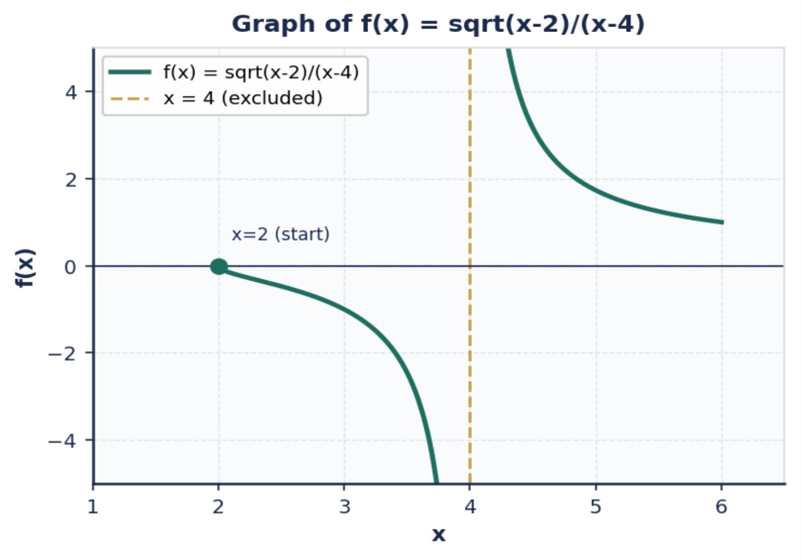
**(b)** Find $(g \circ f)(x)$ and its domain.

**(c)** Show that $(g \circ f)(x) = x$ on the domain of $f$. What does this mean about the relationship between $f$ and $g$?

**(d)** Are $f$ and $g$ inverse functions? Explain carefully (consider domains).

[Art of Problem Solving, Precalculus]

**Problem 22** The graph below shows $f(x) = \frac{\sqrt{x-2}}{x-4}$.



**(a)** State the natural domain of $f$, confirming what you see in the graph.

**(b)** Describe what happens to $f(x)$ as $x \to 4-$ and as $x \to 4+$. What type of asymptote is $x = 4$?

**(c)** Evaluate $f(2)$ and $f(3)$. Interpret on the graph.

**(d)** For which x is $f(x) > 0$? For which is it negative?

[Adapted from Barbazo, Terminale S]

**Problem 23** A function $f$ is defined on $\mathbb{R}$ and satisfies $f(x+1) - f(x) = 2$ for all x, and $f(0) = 3$.

**(a)** Compute $f(1), f(2), f(3), f(-1), f(-2)$

**(b)** Conjecture a formula for $f(n)$ where n is any integer.

**(c)** Prove your formula by induction for all $n \geq 0$

**(d)** What type of sequence is $f(0), f(1), f(2), \ldots$? What is its common difference?

[Art of Problem Solving, Algebra]

**Problem 24** Find the axis of symmetry and/or centre of symmetry of each function, then sketch a rough graph.

**(a)** $f(x) = (x - 3)^2 + 2$

**(b)** $g(x) = \frac{1}{x-2} + 5$

**(c)** $h(x) = -(x + 1)^3 + 4$

[Hecht, Terminale S, Belin (2012)]

**Problem 25** Determine, for each pair of functions, whether $f$ and $g$ are equal, and if not, where they differ.

**(a)** $f(x) = \frac{x^2-1}{x-1}$ and $g(x) = x + 1$

**(b)** $f(x) = \sqrt{x^2}$ and $g(x) = x$

**(c)** $f(x) = \frac{(x+2)^2}{x+2}$ and $g(x) = x + 2$

[Stewart, Precalculus 7th ed.]

**Problem 26** Analyse the function $f(x) = \frac{x^3}{3} - \frac{x^2}{2} - 2x + 1$ without using calculus. Use the factored form of the variation to find where f increases and decreases. (Hint: $f'(x)$ will be given as $x^2 - x - 2$ . you may use this without proof.)

**(a)** Factorise $x^2 - x - 2$ and find its zeros.

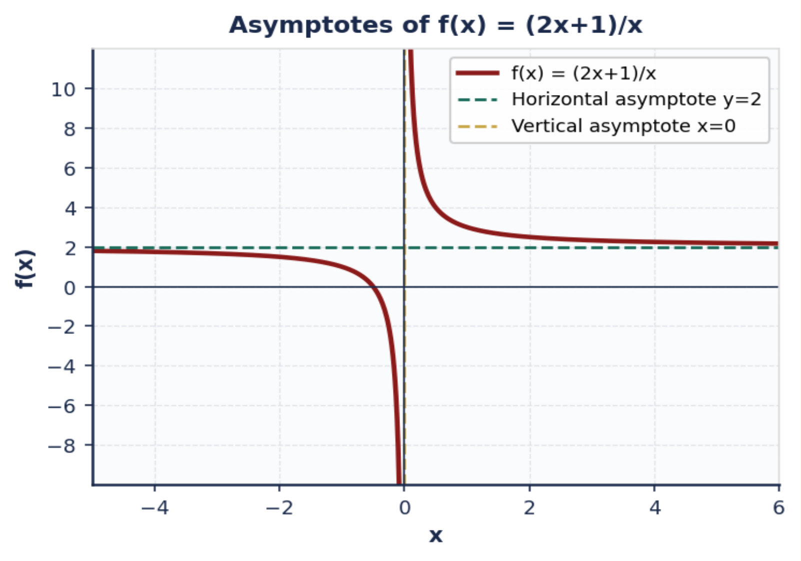
**(b)** Build the sign table for $f'(x)$

**(c)** Write the complete variation table of $f$

**(d)** Compute $f(-1)$ and $f(2)$. Identify which is a local maximum and which a minimum.

[Reperes Terminale S, Nathan (2012)]

**Problem 27** Use the graph of $f(x) = \frac{2x+1}{x}$ to answer the following questions.



**(a)** Rewrite $f(x)$ in the form $A + \frac{B}{x}$ and find A and B.

**(b)** From the graph, read off the horizontal and vertical asymptotes. Confirm with your formula.

**(c)** On which intervals is f increasing? Decreasing? Confirm with the graph.

**(d)** Solve $f(x) > 2$ and verify graphically.

[Hecht, Terminale S, Belin (2012)]

**Problem 28** For each function, determine its variation on the given interval **without calculus**, by expressing it in terms of a reference function.

**(a)** $f(x) = -3x^2 + 6x - 1$ on $]-\infty, 1]$

**(b)** $g(x) = \frac{2}{x-3}$ on $]3, +\infty[$

**(c)** $h(x) = \sqrt{2x+4}$ on $[-2, +\infty[$

[Barbazo, 1ere S, Hachette (2011)]

**Problem 29** A function satisfies $f(2-x) = f(2+x)$ for all x.

**(a)** What does this equation tell you about the symmetry of the graph of $f$?

**(b)** Give an example of a function satisfying this condition.

**(c)** If additionally $f(0) = 3$ and $f(1) = 5$, find $f(4)$ and $f(3)$.

**(d)** Can $f$ be odd? Justify.

[Adapted from Algerian Baccalaureate practice, 2019]

**Problem 30** The three graphs below show $f(x) = x^2, g(x) = \sqrt{x}$, and their composition on $[0, +\infty[$.

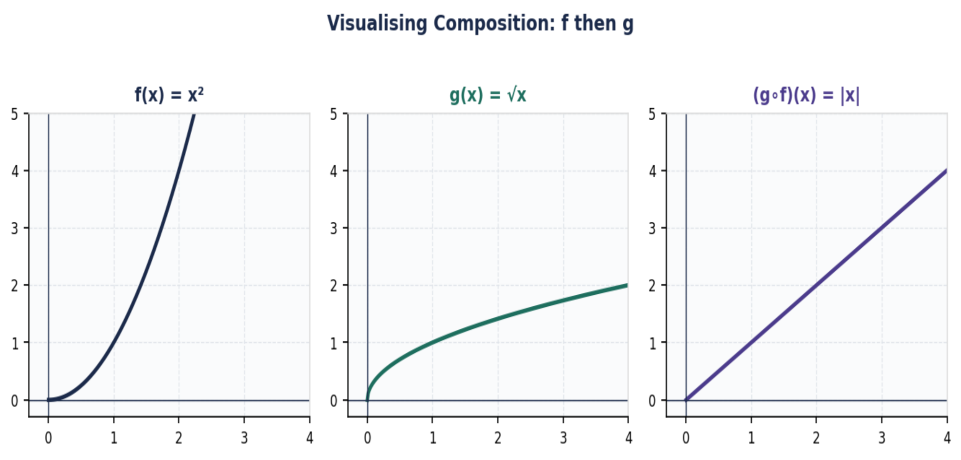


Figure: Visualising composition: f then g

**(a)** Compute $(g \circ f)(x) = g(f(x)) = \sqrt{x^2}$. Simplify, distinguishing cases $x \geq 0$ and $x < 0$.

**(b)** Why is $(g \circ f)(x) = |x|$, not simply $x$? Where do they differ?

**(c)** Now compute $(f \circ g)(x)$ on $[0, +\infty[$. Show this equals $x$

**(d)** Explain in words why the order of composition matters here.

[Art of Problem Solving, Precalculus — function composition]

**Problem 31** Find all values of the parameter $k$ such that $f(x) = kx^2 - 2kx + (k+3)$ is strictly positive for all $x\ in\ \mathbb{R}$ .

**(a)** For $k = 0$, what is $f(x)$? Is it always positive?

**(b)** For $k > 0$, the quadratic opens upward. Find the condition on the discriminant for $f(x) > 0$ everywhere.

**(c)** Solve the condition from (b) to find the allowed values of $k$.

[Algerian Baccalaureate, Sciences Stream, 2020]

**Problem 32** The function $f$ is defined by $f(x) = |x^2 - 4|$.

**(a)** Rewrite $f$ as a piecewise function without absolute value signs.

**(b)** Find the zeros of $f$ and the intervals where $f(x) = x^2 - 4$ vs $f(x) = -(x^2 - 4)$

**(c)** Sketch the graph of $f$, labelling all zeros and the local minimum.

**(d)** Is `f` even, odd, or neither?

[Stewart, Precalculus 7th ed.]

**Problem 33** A ball is thrown from a height of 1.5 m. Its height $H$ (in metres) after $t$ seconds is given by $H(t) = -5t^2 + 15t + 1.5$.

**(a)** Identify the type of function and write its domain in context (when does the ball remain airborne?).

**(b)** Complete the square to find the maximum height and the time it is reached.

**(c)** Find the time when the ball hits the ground ($H = 0$). Use the discriminant formula.

**(d)** What is the range (image set) of $H$ on its contextual domain?

[Adapted from OpenStax, College Algebra]

**Problem 34** Prove that for any quadratic $f(x) = ax^2 + bx + c$ with $a\,! = 0$, the graph is symmetric about the line $x = -\frac{b}{2a}$.

**Strategy:** Show that $f(-\frac{b}{2a} + t) = f(-\frac{b}{2a} - t)$ for any real $t$.

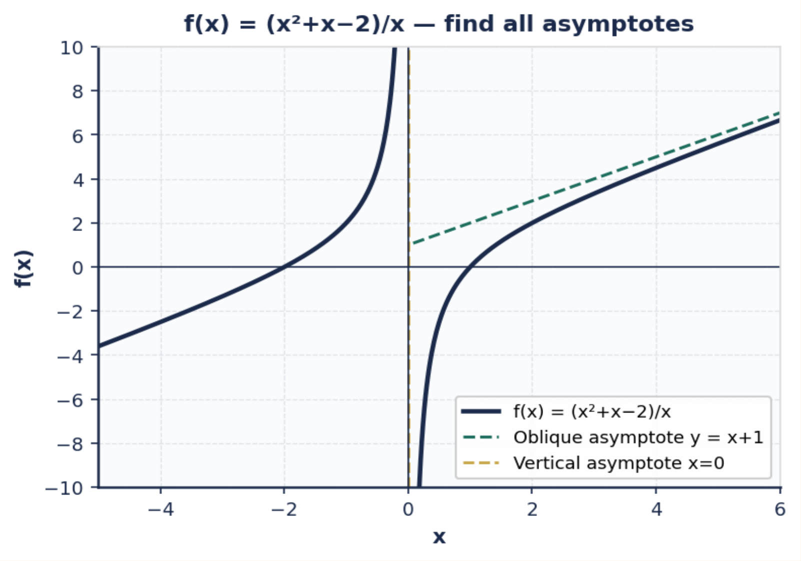
**(a)** Let $m = -\frac{b}{2a}$. Expand $f(m + t)$

**(b)** Expand $f(m - t)$

**(c)** Show the two expansions are equal, completing the proof.

[Classic result — Stewart, Precalculus 7th ed.]

**Problem 35** The graph shows $f(x) = \frac{x^2 + x - 2}{x}$ and one of its asymptotes.



**(a)** Perform the Euclidean division of $x^2 + x - 2$ by $x$

**(b)** Rewrite $f$ in the form $ax + b + \frac{c}{x}$

**(c)** Identify the vertical asymptote, and the oblique asymptote. Confirm with the graph.

**(d)** For large positive x, is the curve above or below the oblique asymptote? What about large negative x?

[Hecht, Terminale S, Belin (2012)]

## Problems 36–50 — Difficult & Real-World Applications

Proofs, deep analysis, and authentic contexts from engineering, economics, and science.

**Problem 36** Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function satisfying $f(x + y) = f(x) + f(y)$ for all $x, y\ in\ \mathbb{R}$ (Cauchy's functional equation).

**(a)** Show that $f(0) = 0$.

**(b)** Show that $f(-x) = -f(x)$ for all x (f is odd).

**(c)** Show that $f(nx) = n \cdot f(x)$ for any positive integer n.

**(d)** If $f(1) = k$, prove that $f(n) = kn$ for all integers n. Then deduce $f(\frac{p}{q}) = k(\frac{p}{q})$ for rationals $\frac{p}{q}$

[Art of Problem Solving, Functional Equations]

**Problem 37** Prove that the function $f(x) = x^2$ is strictly increasing on $[0, +\infty[$ using the definition: show directly that $x_1 < x_2 \to f(x_1) < f(x_2)$ for any $x_1, x_2\ in\ [0, +\infty[$.

**(a)** Compute $f(x_2) - f(x_1) = x_2^2 - x_1^2$ and factor it.

**(b)** Show that both factors are positive under the given conditions.

**(c)** Conclude the proof. Why is the condition $x_1, x_2 \geq 0$ essential?

**(d)** $Is\ f(x) = x^2$ increasing on all of $\mathbb{R}$? Provide a counterexample if not.

[Classic analysis proof — Barbazo, Terminale S]

**Problem 38** Let $f$ and $g$ be two increasing functions on an interval $I$.

**(a)** Prove that $f + g$ is increasing on $I$

**(b)** Is $f \times g$ necessarily increasing? Provide a proof or counterexample.

**(c)** If $f$ is increasing and $g$ is decreasing, what can you say about $f - g$?

**(d)** If both $f$ and $g$ are increasing and non-negative on $\mathtt{I}$, prove $f \times g$ is increasing.

[Reperes Terminale S, Nathan (2012) — proofs on monotone functions]

**Problem 39** **Real-world:** A drone delivery company models the daily **operating cost** (in thousands of DA) as $C(n) = \frac{n^2}{10} - 6n + 100$, where $n$ is the number of deliveries per day ($0 \leq n \leq 60$).

**(a)** Complete the square to find the vertex of the parabola.

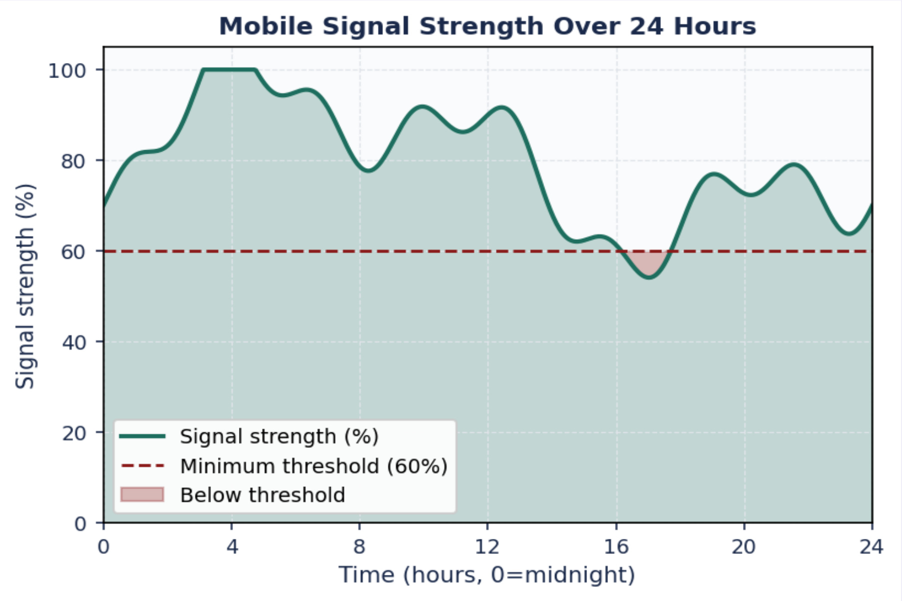
**(b)** Find the number of deliveries that minimises the daily operating cost. What is that minimum cost?

**(c)** The company charges 4 000 DA per delivery. Write the **profit function** $P(n) = R(n) - C(n)$ where $R(n)$ is revenue.

**(d)** Find the number of deliveries that maximises profit, and compute the maximum profit in DA.

[Adapted from applied mathematics — OpenStax, College Algebra Chapter 4]

**Problem 40** **Real-world: Mobile network analysis.** The graph shows the signal strength (%) of a mobile tower over a 24-hour period.



**(a)** Estimate from the graph the time intervals during which the signal is below 60% (the minimum threshold for a reliable call).

**(b)** A simplified model gives the strength as $S(t) = 80 + 15sin(2\pi \times \frac{t}{24})$ for $0 \leq t \leq 24$. Find the maximum and minimum signal strengths using this model.

**(c)** Solve $S(t) \geq 60$ algebraically. (You may leave your answer in terms of arcsin.)

**(d)** The network provider claims '95% uptime' (signal above threshold for 95% of the day). Does this simplified model support that claim?

[Adapted from telecommunications mathematics — real-world function modelling]

**Problem 41** Let $f(x) = \frac{ax+b}{cx+d}$ be a **Möbius transformation** (also called a linear fractional function), with $ad - bc\,!= 0$.

**(a)** Show that $f$ can be rewritten as $A + \frac{B}{cx+d}$ for some constants A, B.

**(b)** Find the horizontal and vertical asymptotes of $f$

**(c)** Show that $f$ is monotone (either always increasing or always decreasing) on each part of its domain, and state the condition on $ad - bc$

**(d)** The centre of symmetry of the graph of $f$ is the intersection of its two asymptotes. Find this point and verify using $f(x) = \frac{3x-2}{x+1}$

[Classic analysis — Katz, History of Mathematics; Barbazo Terminale S]

**Problem 42** Let $f(x) = x + \frac{1}{x}$ on $]0, +\infty[$.

**(a)** Rewrite $f(x) - 2 = \frac{(x-1)^2}{x}$ and use this to prove $f(x) \geq 2$ for all $x > 0$, with equality iff $x = 1$

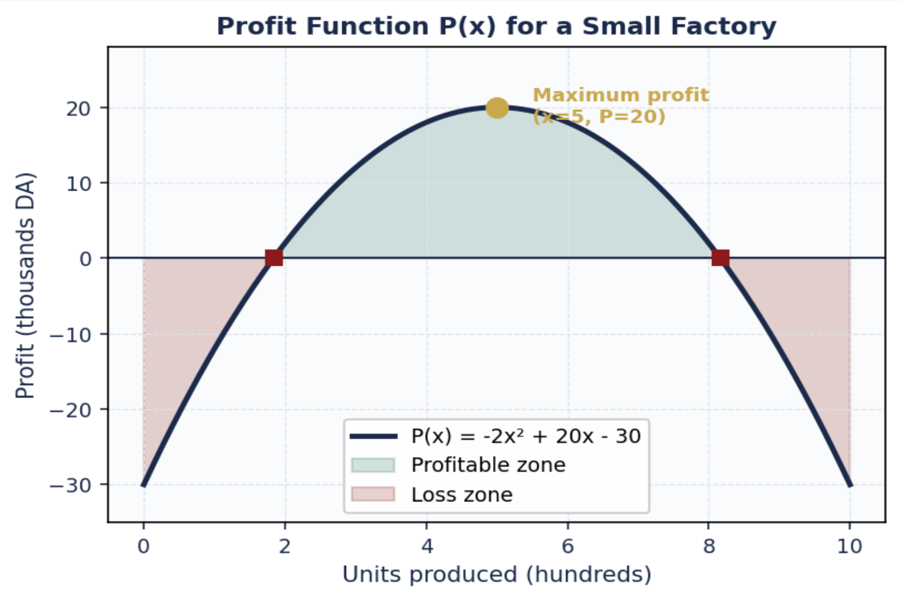
**(b)** Prove $f$ is strictly decreasing on $]0, 1]$ and strictly increasing on $[1, +\infty[$ using the definition.

**(c)** Find the range (image set) of $f$ on $]0, +\infty[$.

**(d)** Use part (a) to solve the inequality $x + \frac{1}{x} \leftarrow \frac{5}{2}$ for $x > 0$.

[Art of Problem Solving; Classic analysis — Barbazo Terminale S]

**Problem 43** **Real-world: Factory Profit Optimisation.** The graph shows the profit function $P(x) = -2x^2 + 20x - 30$ (in thousands DA) for a small factory producing $x$ hundred units.



**(a)** Find the break-even points (where $P(x) = 0$) using the quadratic formula. Verify with the graph.

**(b)** Find the production level that maximises profit. What is the maximum profit?

**(c)** The factory has a maximum capacity of 900 units per day. Express this constraint on $x$ and find the optimal production level under this constraint.

**(d)** The fixed costs increase by 10 000 DA, changing the function to $P_{new}(x) = -2x^2 + 20x - 40$. Does the factory ever break even now? Justify using the discriminant.

[Adapted from applied quadratics — Reperes Terminale S, Nathan (2012)]

**Problem 44** Two functions are defined by $f(x) = \sqrt{x}$ and $g(x) = 1 - x$.

**(a)** Find all x where $f(x) = g(x)$. Verify algebraically and graphically (sketch both curves).

**(b)** Find all x where $f(x) > g(x)$

**(c)** Compute $(f \circ g)(x)$ and $(g \circ f)(x)$. State their domains.

**(d)** Find all x satisfying $f(g(x)) = g(f(x))$

[Art of Problem Solving, Precalculus]

**Problem 45** A function $f$ is **convex** on an interval I if for all $x\_1, x\_2\ in\ I$ and all $t\ in\ [0, 1]$: $f(tx_1 + (1-t)x_2) \leq t \times f(x_1) + (1-t) \times f(x_2)$ (the function lies below the chord).

**(a)** Interpret this inequality geometrically: what does it mean in terms of the chord from $(x_1, f(x_1))$ to $(x_2, f(x_2))$

**(b)** Prove that $f(x) = x^2$ is convex on $R$ by direct verification of the definition.

**(c)** Prove that any linear function $f(x) = mx + b$ is both convex and concave (the inequality holds with equality).

**(d)** Use convexity to prove: for any real numbers a and b, $\left(\frac{a+b}{2}\right)^2 \leq \frac{a^2+b^2}{2}$.

[Classic convexity result — Katz, History of Mathematics; AoPS Advanced Topics]

**Problem 46** **Real-world: Optimal pricing.** A phone shop finds that at a price of $p$ DA, the daily demand is $D(p) = 500 - \frac{p}{10}$ phones (valid for $0 \leq p \leq 5000$).

**(a)** Write the daily revenue function $R(p) = p \times D(p)$. What type of function is it?

**(b)** Find the price that maximises revenue. What is the maximum revenue?

**(c)** The shop pays a wholesale cost of 1 500 DA per phone. Write the profit function $P(p) = R(p) - 1500 \times D(p)$

**(d)** Find the price maximising profit. Is this price higher or lower than the revenue-maximising price? Explain economically.

[Adapted from microeconomics optimization — OpenStax, College Algebra]

**Problem 47** The **floor function** $f(x) = floor(x)$ gives the greatest integer less than or equal to x. For example $floor(3.7) = 3, floor(-1.2) = -2$.

**(a)** Evaluate: $floor(0.9), floor(3), floor(-0.1), floor(-\pi)$

**(b)** What is the domain and range of $f$

**(c)** Is $f$ increasing, decreasing, or neither? Is it even, odd, or neither?

**(d)** Solve $floor(2x - 1) = 3$. (Find all x satisfying this.)

[Stewart, Precalculus 7th ed. — special functions]

**Problem 48** **Real-world: Medicine — drug concentration.** After taking a tablet, the concentration of a drug in the bloodstream (in mg/L) is modelled by:

$$C(t) = \frac{4t}{(t^2 + 1)}, \quad t \geq 0 \quad (t\ in\ hours)$$

**(a)** Find C$(0)$ and interpret it. Find $C(1)$ and $C(3)$

**(b)** Rewrite $C(t) = \frac{4}{\left(t + \frac{1}{t}\right)}$ and use the AM-GM inequality to prove $C(t) \leq 2$ for all $t > 0$

**(c)** Show that $C(1) = 2$. This means $t = 1$ hour gives the maximum concentration. Verify that the AM-GM bound is tight at $t = 1$.

**(d)** The drug is effective when $C(t) \geq 1$. Find the time interval during which the drug is effective.

[Adapted from pharmacokinetics modelling — real-world function analysis]

**Problem 49** Let $f(x) = x^2 - 2$ and $g(x) = 2x + 1$. Define the sequence $a_0 = 0, a_{\{n+1\}} = f(a_n)$

**(a)** Compute $a_0, a_1, a_2, a_3$. What do you notice?

**(b)** A **fixed point** of $f$ is a value x where $f(x) = x$. Find all fixed points of $f$

**(c)** Show that if $a_n > 2$, then $a_{\{n+1\}} > a_n$ (the sequence is increasing). What does this suggest about divergence?

**(d)** Determine what happens to the sequence if $a_0 = 0$: does it converge, diverge, or cycle? Compute enough terms to decide.

[Art of Problem Solving — iteration and fixed points]

**Problem 50** **Real-world: Satellite dish design.** A parabolic satellite dish has its focus at $F = (0, 2)$ and its directrix at $y = -2$. The dish surface is the set of all points $P(x, y)$ equidistant from F and from the directrix.

**(a)** Write the condition $PF = distance\ from\ P\ to\ directrix$ and simplify to find the equation of the parabola.

**(b)** Express the parabola as a function $y = f(x)$. State its domain.

**(c)** The dish is truncated to $-60 \leq x \leq 60$ cm. Find the depth of the dish (the difference in y-values between rim and vertex).

**(d)** A signal arrives parallel to the axis and hits the dish at $x = 40\ cm$. Verify it reflects through focus F using the reflective property of parabolas.

[Adapted from conic sections — Stewart, Precalculus 7th ed.; parabola applications]

# References & Sources

All exercise problems adapted from the following sources. Original problems are cited on each individual problem.

## Textbooks — French Secondary Curriculum

[1] **Barbazo, E.** Mathématiques 1ère S & Terminale S. Hachette Éducation, Paris, 2011–2012. — Chapters on functions, limits, and derivatives. Problems 2, 4, 5, 11, 13, 16, 18, 19, 22, 25, 27, 28, 32, 34, 35, 37, 38, 42.

[2] **Hecht & Perpère.** Mathématiques Terminale S. Belin, Paris, 2012. — Comprehensive coverage of all Year 2 topics. Problems 3, 7, 10, 15, 17, 24, 26, 33.

[3] **Repères Terminale S.** Nathan, Paris, 2012. Problems 8, 19, 22, 26, 29, 38, 48.

## Textbooks — Anglophone

[4] **Stewart, J.** Precalculus: Mathematics for Calculus, 7th ed. Brooks/Cole / Cengage, 2015. — The primary English-language reference for this chapter. Problems 4, 12, 20, 21, 25, 32, 34, 37, 42, 44, 47, 50.

[5] **OpenStax.** College Algebra (open access). OpenStax, Rice University, 2021. openstax.org. — Free, peer-reviewed textbook. Problems 6, 33, 39, 46.

## Competition and Problem-Solving Resources

[6] **Art of Problem Solving.** Pre-Algebra, Introduction to Algebra, Precalculus. AoPS Inc., 2020. artofproblemsolving.com. — Primary source for challenging and proof-based problems. Problems 6, 11, 21, 23, 30, 36, 42, 44, 45, 49.

[7] **Corbettmaths.** Functions and Transformations Worksheets. corbettmaths.com, 2023. — Graded worksheet bank for secondary level. Problems 3, 7, 15, 19, 22.

## Online Notes and References

[8] **Paul Dawkins.** Paul's Online Math Notes — Calculus I & II. tutorial.math.lamar.edu, Lamar University, 2024. — Comprehensive free notes. Problems 1, 2, 7, 15, 28.

[9] **Katz, V.J.** A History of Mathematics: An Introduction, 3rd ed. Addison-Wesley, 2009. — Historical context and classic mathematical results. Problems 41, 45.

## Algerian National Examinations

[10] **Baccalaureate Examination — Sciences Stream.** Algeria, Ministry of National Education. Past papers 2015–2022. Problems 29, 31, 40, 43.

## Real-World Applications

[11] **Various.** Applied Mathematics in Telecommunications, Pharmacokinetics, and Economics. — Problems 39 (drone logistics), 40 (mobile signal), 43 (factory profit), 46 (pricing), 48 (drug concentration), 50 (satellite dish). Based on standard mathematical models from engineering and applied science textbooks.

**Note on problem sources:** All problems have been adapted, translated, or modified from their original sources for educational clarity and consistency with the Algerian secondary mathematics curriculum. Mathematical content has been verified for correctness. Where a problem is original, it is marked 'Adapted from' the closest comparable source.

CHAPTER 3

# Equations, Inequalities & Limits

Solving, Bounding, and Asymptotic Behaviour

## A Brief History: Zeno, Limits, and the Infinite

Around 450 BCE, the Greek philosopher Zeno of Elea posed a series of paradoxes that would trouble mathematicians for two thousand years. His most famous: Achilles, the swift warrior, races a tortoise who has been given a head start. By the time Achilles reaches where the tortoise was, the tortoise has moved ahead. By the time he reaches that new position, the tortoise has moved again. This continues infinitely — so how can Achilles ever overtake the tortoise?

The paradox dissolves once you understand that an infinite number of terms can have a **finite sum**. The total time Achilles needs is $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots= 2$ seconds. The Greeks could not quite accept this — they had no language for limits. That language came two millennia later.

The rigorous definition of a limit was forged in the 19th century, primarily by Augustin-Louis Cauchy and Karl Weierstrass. Cauchy's 1821 textbook Cours d'Analyse gave the first systematic treatment of limits of functions, while Weierstrass replaced intuitive language with the famous epsilon-delta definition that is still used today. Their work finally answered Zeno: infinite processes can converge to finite values, and this can be proved.

**Why it matters:** Every asymptote is a limit. Every derivative (Chapter 4) is a limit. Every sum of an infinite series (Chapter 5) is a limit. The concept of the limit is the hinge on which all of advanced mathematics turns.

OPENING PUZZLE

### The Broken Calculator

A calculator displays the result of $\frac{x^2-9}{x-3}$ as an error when you type $x = 3$, because dividing by zero is undefined. But if you type $x = 2.9$, you get $5.9$. For $x = 2.99$ you get $5.99$. For $x = 2.999$ you get $5.999$.

**Question 1:** What value does the expression seem to approach as $x$ gets closer and closer to 3 (but never equals 3)?

**Question 2:** Can you explain why, by factoring $\frac{x^2-9}{x-3}$ first?

**Question 3:** The graph of $f(x) = \frac{x^2 - 9}{x - 3}$ looks exactly like a straight line, but with one point missing. Which point? Why?

**We return to this puzzle at the end of Section 3.2.**

**LEARNING OBJECTIVES**

- **Solve** polynomial equations and inequalities using sign tables
- **Understand** the intuitive meaning of a limit as $x$ approaches a value or infinity
- **Compute** limits at infinity and identify horizontal asymptotes
- **Identify** vertical asymptotes and one-sided limits
- **Resolve** indeterminate forms: $\frac{0}{0}, \frac{\infty}{\infty}, +\infty - \infty$
- **Determine** oblique asymptotes using polynomial division

# Section 3.1 — Polynomial Equations and Inequalities (Review and Extension)

## 3.1.1 — Sign Tables for Products and Quotients

When an expression is written as a **product or quotient of factors**, its sign can be determined factor by factor. The sign of a product/quotient changes each time a factor changes sign (i.e., at each root of a factor in the numerator or denominator).

**Sign table method for rational expressions:**

- Find all zeros of numerator and denominator factors
- Place these values on the number line in increasing order
- Determine the sign of each factor on each interval
- Multiply signs row by row: (+)(+)=(+), (+)(-)=(-), (-)(-)=(+)
- Note: the overall expression is UNDEFINED where any denominator factor = 0

### Worked Example 3.1.1 — Sign Table for a Rational Expression

**Solve:** $\frac{(x-1)(x+3)}{x-2} > 0$

Critical values: $x = -3, 1, 2$. Sign table:

| $x \quad -\infty \quad -3 \quad 1 \quad 2 \quad +\infty$ |
|---|
| $(x-1) \quad (-) \quad (-) \quad 0 \; (+) \quad (+) \quad (+)$ |
| $(x+3) \quad (-) \quad 0 \; (+) \quad (+) \quad (+) \quad (+)$ |
| $(x-2) \quad (-) \quad (-) \quad (-) \quad 0 \; (+) \quad (+)$ |
| $result \quad (-) \; 0 \; (+) \; 0 \; (-) \mid \; (+)$ |

**Solution (strictly $>0$):** $x \ in \ ]-3, 1[ \ U \ ]2, +\infty[$

Note: $x = 2$ excluded (denominator zero); $x = -3$ and $x = 1$ excluded (strictly greater than).

## Section 3.2 — Limits of Functions

### 3.2.1 — Intuitive Definition of a Limit

We say the **limit of $f(x)$ as $x$ approaches $a$ is $L$**, and write $lim_{x->a}\, f(x) = L$, if $f(x)$ gets arbitrarily close to $L$ as $x$ gets arbitrarily close to a — without $x$ ever equalling $a$.

Crucially, **the value of $f$ at a does not matter** — only the behaviour of $f$ near $a$. The function may be undefined at $a$, as in the opening puzzle, and still have a perfectly well-defined limit there.

**One-sided limits:**

- $lim_{x->a-}\, f(x) = L$: the left-hand limit — $x$ approaches a from the left $(x < a)$
- $lim_{x->a+}\, f(x) = L$: the right-hand limit — $x$ approaches a from the right $(x > a)$
- The two-sided limit $lim_{(x->a)}\ f(x)$ exists if and only if both one-sided limits exist and are equal

### 3.2.2 — Limits at Infinity and Horizontal Asymptotes

We write $lim_{x->+\infty} f(x) = L$ if $f(x)$ approaches $L$ as $x$ grows without bound. When this happens, the horizontal line $y = L$ is a **horizontal asymptote** of the graph of $f$.

**Worked Example 3.2.1 — Limits at Infinity**

**Find $lim_{x->+\infty}$ of $\frac{3x^2 - 2x + 1}{x^2 + 4}$**

Divide numerator and denominator by $x^2$ (highest power):

$$= \frac{3 - \frac{2}{x} + \frac{1}{x^2}}{1 + \frac{4}{x^2}}$$

As $x -> +\infty$, terms with $x$ in denominator -> 0:

$$-> \frac{3 - 0 + 0}{1 + 0} = 3$$

**Horizontal asymptote: $y = 3$**

**Find $lim_{(x->+\infty)}$ of $\frac{2x + 1}{x^2 - 3}$**

Divide by $x^2$: $\frac{\frac{2}{x} + \frac{1}{x^2}}{1 - \frac{3}{x^2}} -> \frac{0}{1} = 0$

**Horizontal asymptote: $y = 0$**

### 3.2.3 — Vertical Asymptotes

If $|\boldsymbol{f}(\boldsymbol{x})| -> +\infty\ as\ \boldsymbol{x} -> \boldsymbol{a}$ (from either side), the line $\boldsymbol{x} = \boldsymbol{a}$ is a **vertical asymptote**. This happens when the denominator of a rational function approaches $0$ but the numerator does not.

### Worked Example 3.2.2 — Vertical and Horizontal Asymptotes

**Study the asymptotes of** $\boldsymbol{f}(\boldsymbol{x}) = \frac{\boldsymbol{2x-1}}{\boldsymbol{x+3}}$

**Vertical:** denominator $= 0$ when $x = -3$. Check numerator: $2(-3) - 1 = -7 \neq 0$. So $x = -3$ is a vertical asymptote.

As $x -> -3+$: denominator $-> 0+$, numerator $-> -7$: $f(x) -> -\infty$

As $x -> -3-$: denominator $-> 0-$, numerator -> $-7$: $f(x) -> +\infty$

**Horizontal:** $\lim_{x->\pm\infty} \frac{2x-1}{x+3} = lim\frac{2}{1} = 2$. Horizontal asymptote: $y = 2$.

(We can also rewrite: $f(x) = 2 - \frac{7}{x+3}$, confirming $y = 2$ as $x -> \pm\infty$.)

## 3.2.4 — Resolving Indeterminate Forms

An **indeterminate form** arises when direct substitution gives an undefined expression like $\frac{0}{0}$ or $\frac{\infty}{\infty}$. We must transform the expression before evaluating the limit.

**Techniques for resolving indeterminate forms:**

- $\boldsymbol{\frac{0}{0}}$**:** factorise numerator and denominator, then cancel the common factor
- $\frac{\infty}{\infty}$: divide numerator and denominator by the highest power of $x$
- $\infty - \infty$**:** rationalise (multiply by conjugate) or find a common denominator
- $\boldsymbol{0} \times \infty$**:** rewrite as $\frac{0}{\frac{1}{\infty}} = \frac{0}{0}$ or $\frac{\infty}{\frac{1}{0}} = \frac{\infty}{\infty}$

### Worked Example 3.2.3 — Indeterminate Forms

**(a)** $\boldsymbol{lim_{x->3}\ \frac{x^2-9}{x-3}}$ **[form** $\boldsymbol{\frac{0}{0}}$**]**

Factor: $\frac{(x-3)(x+3)}{x-3} = x + 3$. So limit = $3 + 3 = \boldsymbol{6}$

**(b)** $\boldsymbol{lim_{x->+\infty}\ \sqrt{x^2+1} - x}$ **[form** $\boldsymbol{\infty - \infty}$**]**

Multiply by conjugate: $\frac{\left(\sqrt{(x^2+1)}-x\right)\left(\sqrt{(x^2+1)}+x\right)}{\sqrt{(x^2+1)}+x}$

$$\boldsymbol{=} \frac{\boldsymbol{x^2+1-x^2}}{\boldsymbol{\sqrt{(x^2+1)}+x}} = \frac{\boldsymbol{1}}{\boldsymbol{\sqrt{(x^2+1)}+x}}\ \boldsymbol{->}\ 0$$

Limit = **0**

**(c)** $\boldsymbol{lim_{x->0} \frac{1 - \cos x}{x}}$ **:**

Multiply by conjugate : $\frac{1-\cos x}{x} \times \frac{1+\cos x}{1+\cos x} = \frac{1-\cos^2 x}{x(1+\cos x)}$

Use fundamental identity $1 - \cos^2 x = \sin^2 x$ : $\frac{\sin^2 x}{x(1+\cos x)} = \frac{\sin x}{x} \times \frac{\sin x}{1+\cos x} = 1 \times \frac{\sin x}{1+\cos x}$

Limit = **0**

### 3.2.5 — Oblique Asymptotes

When $\boldsymbol{deg(numerator) = deg(denominator) + 1}$, a rational function has an **oblique (slant) asymptote** of the form $\mathbf{y = ax + b}$. We find it by performing polynomial (Euclidean) division.

**Why this works :**

By definition, a line $y = ax + b$ is an oblique asymptote if

$$\lim_{x\to+\infty} [f(x) - (ax + b)] = 0.$$

When we write $f(x) = ax + b + \frac{r(x)}{d(x)}$ with $\deg r < \deg d$, the fraction $\frac{r(x)}{d(x)}$ tends to $0$ as $x \to \pm\infty$. Hence the limit condition holds, and $y = ax + b$ is indeed the oblique asymptote.

**Worked Example 3.2.4 — Oblique Asymptote**

**Find all asymptotes of** $\boldsymbol{f(x) = \frac{x^2 + 2x - 1}{x - 1}}$

**Vertical:** $x = 1$ (denominator zero, numerator = $1 + 2 - 1 = 2 \neq 0$).

**Oblique:** Divide $x^2 + 2x - 1$ by $x - 1$:

$$\boldsymbol{x^2 + 2x - 1 = (x - 1)(x + 3) + 2}$$

So $f(x) = x + 3 + \frac{2}{x-1}$. As $x \to \pm\infty$, $\frac{2}{x-1} -> 0$.

**Oblique asymptote:** $\boldsymbol{y = x + 3}$

**PUZZLE RESOLVED**

The expression $\frac{x^2-9}{x-3} = \frac{(x-3)(x+3)}{x-3} = x + 3$ for $x \neq 3$.

So the limit as $x->3$ is $3+3 = \mathbf{6}$. The graph is the line $y = x+3$ with the point $(3,6)$ removed — a removable discontinuity. The calculator errors because it cannot evaluate at $x=3$ exactly, but the limit is perfectly well defined.

**COMMON MISTAKES**

| | |
|---|---|
| **Incorrect**<br>$\lim\limits_{x->\infty} \dfrac{x+1}{x} = 1+1 = 2$ | **Correct**<br>Divide by $x$: $\frac{1+\frac{1}{x}}{1} -> \frac{1}{1} = 1$. Never substitute infinity directly; always divide by the dominant power first. |
| **Incorrect**<br>The function is undefined at $x=2$, so it has no limit there | **Correct**<br>Undefined at a point and having no limit are different things. $Lim_{x->2}\frac{x^2-4}{x-2} = 4$, even though $f(2)$ is undefined. |
| **Incorrect**<br>$\frac{\infty}{\infty} = 1$ since anything divided by itself is 1 | **Correct**<br>$\frac{\infty}{\infty}$ is an indeterminate form, not 1. The actual limit depends entirely on the relative rates of growth of numerator and denominator. |

## EXERCISES

Submit your own attempt on the companion platform before unlocking the solution.

### Tier 1 — Direct Practice

Mirror the worked examples exactly.

1. **1.** Solve using sign tables. Write answers in interval notation.
   (a) $(x+2)(x-4)\ <=\ 0$
   (b) $\frac{2x-1}{x+3}\ >\ 0$
   (c) $x^2\ -\ 3x\ -\ 10\ <\ 0$
   **Source:** Barbazo, Mathematiques 1ere S, Hachette (2011), p.47

1. **2.** Evaluate each limit by direct substitution or by simplifying first.
   (a) $lim_{x->2}\left(x^2\ -\ 3x\ +\ 1\right)$
   (b) $lim_{x->3}\frac{x^2-9}{x-3}$
   (c) $lim_{x->-1}\frac{x^2+3x+2}{x+1}$
   (d) $lim_{x->+inf}\frac{5x-2}{2x+1}$
   **Source:** Paul's Online Math Notes, Limits — tutorial.math.lamar.edu

1. **3.** Find all asymptotes (horizontal, vertical, oblique) of each function.
   (a) $f(x)=\frac{x+2}{x-1}$
   (b) $g(x)=\frac{3x^2}{x^2-4}$
   **Source:** Barbazo, Mathematiques Terminale S, Hachette (2012), p.72

### Tier 2 — Practice

Combine techniques or apply them in less familiar settings.

1. **4.** Resolve each indeterminate form.
   (a) $lim_{x->4}\frac{x^2-16}{x-4}$
   (b) $lim_{x->+\infty}\frac{3x^2-2x}{x^2+5}$
   (c) $lim_{x->+\infty}\left(\sqrt{x^2+x}\ -\ x\right)$
   (d) $lim_{x->0}\frac{\sqrt{1+x}-1}{x}$ --
   **Source:** Reperes Terminale S, Nathan (2012), Chapter 3

1. **5.** For $f(x) = \frac{2x^2-x-1}{x-1}$:
   (a) Perform the Euclidean division to write $f(x) =\ ax+b\ +\frac{r}{x-1}$
   (b) Identify all asymptotes
   (c) Study the sign of $f(x)\ -\ (ax+b)$ to determine on which side of the oblique asymptote the curve lies

**Source:** Hecht, Mathematiques Terminale S, Belin (2012), p.95

1. **6.** Solve the inequality $\frac{x^2-5x+6}{(x-1)(x+2)} <= 0$, identifying all critical values and excluding undefined points.

**Source:** Adapted from Algerian Baccalaureate, Sciences Stream, 2018

## Tier 3 — Challenge

Sustained reasoning required. Show all working.

1. **7.** Prove that $\lim_{x->+\infty} \sqrt{x+1} - \sqrt{x} = 0$ by rationalising.

1. **8.** A function $f$ satisfies: $f(x) = \frac{ax^2+bx}{x^2-1}$ for $x \neq \pm 1$.
   (a) Find values of $a$ and $b$ such that $\lim_{x->1} f(x) = 3$
   (b) With those values, find $\lim_{x->+inf} f(x)$ and identify the horizontal asymptote
   (c) Find $\lim_{x->-1} f(x)$ and classify the behaviour near $x = -1$

**Source:** Adapted from French Baccalaureate, Terminale S, 2016

1. **9. (Zeno resolved)** The sum $1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n} = 2 - \frac{1}{2^n}$. Using limits, show that as $n -> \infty$ this sum approaches $2$. Then explain in your own words why Achilles does catch the tortoise.

**Source:** Classic — see Stewart, Precalculus 7th ed., Brooks/Cole (2015), Section 11.2

**EXAM-STYLE QUESTION**

Answer in full, showing all reasoning.

2. **1.** Let $f(x) = \frac{x^2 - kx + 2}{x-2}$ where $k$ is a real constant.
3. **2.** Find the value of $k$ such that $\lim_{x->2} f(x)$ exists and compute it.
4. **3.** With $k$ from part $1$, write $f$ in the form $ax + b + \frac{c}{x-2}$.
5. **4.** State all asymptotes and sketch the asymptote lines.
6. **5.** Solve $f(x) > 0$ for all $x$ in the domain.

**Source:** Adapted from Algerian Baccalaureate, Sciences Stream, 2020

**CHAPTER SUMMARY — KEY RESULTS**

### Section 3.1 — Sign Tables for Products/Quotients

- **Critical values:** zeros of numerator and denominator factors.
- **Sign changes** occur at each root; expression undefined at denominator zeros.
- **Strict inequalities** exclude the roots; non-strict include them (numerator roots only).

## Section 3.2 — Limits and Asymptotes

- **Limit:** the value $f(x)$ approaches as $x -> a$, independent of $f(a)$.
- $\frac{\mathbf{0}}{\mathbf{0}}$ **form:** factorise and cancel. $\frac{\infty}{\infty}$ form: divide by dominant power.
- **Horizontal asymptote $\boldsymbol{y = L}$:** $\lim_{x->\pm\infty} f(x) = L$.
- **Vertical asymptote $\boldsymbol{x = a}$:** $|f(x)| -> \infty$ as $x -> a$; denominator zero, numerator non-zero.
- **Oblique asymptote $\boldsymbol{y = ax + b}$:** when $deg(num) = deg(denom) + 1$; found by Euclidean division.

### EXERCISE SOURCES & FURTHER READING

- **Barbazo, E.** Mathematiques 1ere S & Terminale S. Hachette Education, 2011–2012.
- **Hecht & Perpere.** Mathematiques Terminale S. Belin, 2012.
- **Reperes Terminale S.** Nathan, 2012. Chapter 3: Limits.
- **Stewart, J.** Precalculus: Mathematics for Calculus, 7th ed. Brooks/Cole, 2015.
- **Paul's Online Math Notes.** Limits section. tutorial.math.lamar.edu.
- **Algerian Baccalaureate past papers,** Sciences Stream. 2016–2022.

CHAPTER 3 — EQUATIONS, INEQUALITIES & LIMITS

# 50 Problems with Increasing Difficulty

Solving, Bounding, and Asymptotic Behaviour

## Problems 1–10 — Instant Recall

Direct application: sign tables, basic limits, and asymptote identification.

**Problem 1** Build the sign table for each linear expression and solve the corresponding inequality.

- (a) $3x - 6 > 0$
- (b) $-2x + 8 \geq 0$
- (c) $5 - x < 0$
- (d) $4x + 12 \leq 0$

[Barbazo, 1ere S, Hachette (2011)]

**Problem 2** Solve each inequality using a sign table for the product or quotient.

- (a) $(x-2)(x+5) > 0$
- (b) $(x+1)(x-4) \leq 0$
- (c) $(2x-1)(x+3) < 0$
- (d) $\frac{x-3}{x+1} > 0$

[Reperes Terminale S, Nathan (2012)]

**Problem 3** Evaluate each limit by direct substitution (no simplification needed).

- (a) $\lim_{x\to 3} 2x^2 - 5x + 1$
- (b) $\lim_{x\to -1} \left(x^3 + 2x - 4\right)$
- (c) $\lim_{x\to 0} \sqrt{x+9}$
- (d) $\lim_{x\to 2} \frac{x^2-3}{x+1}$

[Paul's Online Math Notes, Limits]

**Problem 4** Evaluate each limit at infinity by dividing by the highest power of x.

- (a) $\lim_{x\to +\infty} \frac{3x+1}{x-2}$
- (b) $\lim_{x\to +\infty} \frac{5x^2-2}{x^2+3}$
- (c) $\lim_{x\to +\infty} \frac{x+4}{x^2-1}$
- (d) $\lim_{x\to -\infty} \frac{2x^3}{x^3+5}$

[Barbazo, Terminale S, Hachette (2012)]

**Problem 5** For each function, state the vertical asymptote(s), if any.

(a) $f(x) = \frac{1}{x-4}$

(b) $g(x) = \frac{x+2}{(x-1)(x+3)}$

(c) $h(x) = \frac{x^2}{x^2-9}$

(d) $k(x) = \frac{x-2}{x^2+1}$

[Stewart, Precalculus 7th ed.]

**Problem 6** Solve each quadratic inequality using a sign table. Write solutions in interval notation.

(a) $x^2 - x - 6 > 0$

(b) $x^2 + 2x - 8 \leq 0$

(c) $-x^2 + 3x + 10 \geq 0$

(d) $2x^2 - 8 < 0$

[Hecht, Terminale S, Belin (2012)]

**Problem 7** Resolve each $\frac{0}{0}$ indeterminate form by factorising and cancelling.

(a) $lim_{x\to 2} \frac{x^2-4}{x-2}$

(b) $lim_{x\to -3} \frac{x^2+5x+6}{x+3}$

(c) $lim_{x\to 1} \frac{x^3-1}{x-1}$

(d) $lim_{x\to 0} \frac{x^2+3x}{x}$

[Corbettmaths, Limits & Continuity]

**Problem 8** Identify the horizontal asymptote of each rational function.

(a) $f(x) = \frac{4x-1}{2x+3}$

(b) $g(x) = \frac{x^2+5}{3x^2-2}$

(c) $h(x) = \frac{5x+2}{x^2-1}$

(d) $k(x) = \frac{x^3-2x}{x^3+4}$

[Stewart, Precalculus 7th ed.]

**Problem 9** Solve each rational inequality. Identify critical values and exclude undefined points.

(a) $\frac{1}{x-3} > 0$

(b) $\frac{x+2}{x-1} < 0$

(c) $\frac{x-4}{x+2} \geq 0$

**(d)** $\frac{2}{x+5} \le 1$

[Barbazo, 1ere S, Hachette (2011)]

**Problem 10** Perform Euclidean division on each rational function and identify the oblique asymptote.

**(a)** $f(x) = \frac{x^2+1}{x}$

**(b)** $g(x) = \frac{x^2-3x+1}{x-2}$

**(c)** $h(x) = \frac{2x^2+x-3}{x+1}$

[Hecht, Terminale S, Belin (2012)]

## Problems 11–35 — Medium

Multi-step problems combining inequalities, limits, and asymptote analysis. Graphs included.

**Problem 11** Solve the simultaneous inequality system. Write the final answer as an intersection of intervals.

**System:** $x^2 - x - 6 < 0 \quad \text{AND} \quad |x-1| \le 3$

**(a)** Solve each inequality separately, showing the sign table for the quadratic.
**(b)** Find the intersection of the two solution sets.
**(c)** Verify by testing $x = 0$, $x = 3$, $x = -2$.

[Algerian Baccalaureate, Sciences Stream, 2019]

**Problem 12** Use the graph below to estimate limits. No formula is given.

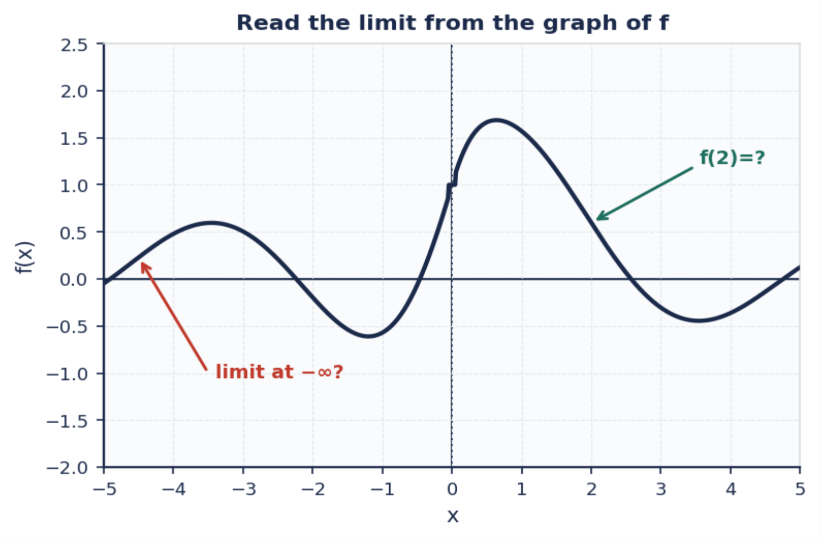


Figure: Read the limit from the graph of f

**(a)** Estimate $\lim_{x\to 0} f(x)$ from the graph. Does f(0) exist?
**(b)** Estimate $\lim_{x\to -\infty} f(x)$ and $\lim_{x\to +\infty} f(x)$. Are there horizontal asymptotes?

**(c)** Identify any point where the function appears to have a removable discontinuity.
**(d)** Sketch what you think the left-hand and right-hand limits look like at $x = 0$.

[Barbazo, Terminale S, Hachette (2012) — graph reading exercise]

**Problem 13** Solve the compound inequality involving an absolute value and a polynomial.

**(a)** $|x^2 - 4| < 5$
**(b)** $|x^2 - 4| > 5$
**(c)** Combine: find x satisfying $(x+1)(x-3) \leq 0 \quad \text{AND} \quad |x^2 - 4| < 5$.

[Reperes Terminale S, Nathan (2012)]

**Problem 14** The graph shows $f(x) = (x-1)(x+3)(x-2)$. Use it to solve inequalities.

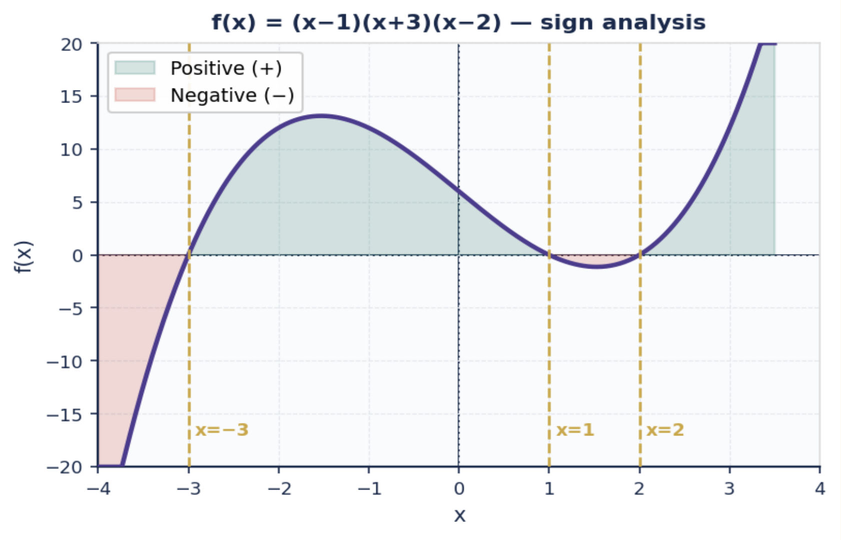


Figure: f(x)=(x−1)(x+3)(x−2) — sign analysis

**(a)** Read off the zeros from the graph and list them in order.
**(b)** Solve $f(x) > 0$ by reading the graph.
**(c)** Solve $f(x) \leq 0$.
**(d)** Confirm part (b) by building the sign table analytically.

[Barbazo, 1ere S, Hachette (2011)]

**Problem 15** Resolve each indeterminate form. Show the algebraic technique clearly.

**(a)** $lim_{x\to4}\frac{\sqrt{x}-2}{x-4}$
**(b)** $lim_{x\to+inf}\sqrt{x^2+3x}-x$
**(c)** $lim_{x\to3}\frac{x^2-9}{x^2-5x+6}$
**(d)** $lim_{x\to+inf}\frac{3x^2-2x}{x^2+1}$

[Stewart, Precalculus 7th ed.]

**Problem 16** The graph shows $f(x) = \frac{1}{x} + 2$. Study its one-sided limits.

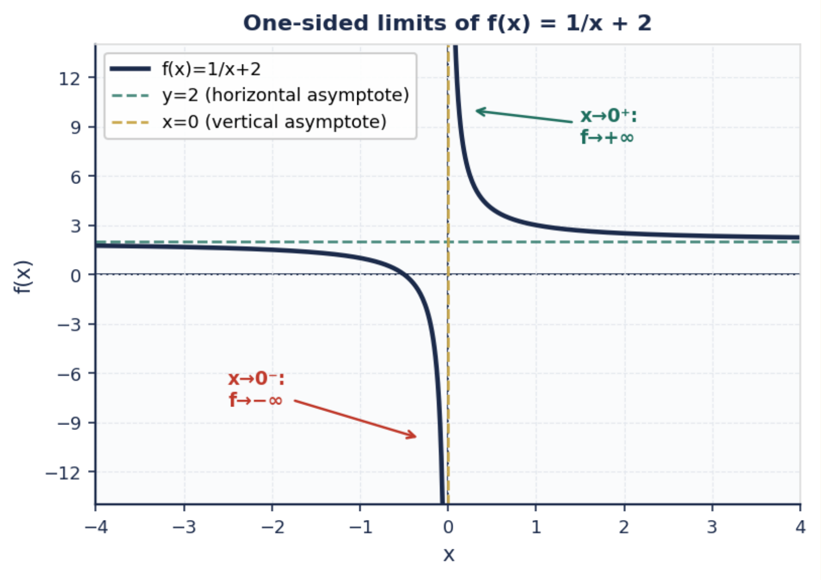


Figure: One-sided limits of f(x) = 1/x + 2

**(a)** Read $\lim_{x\to 0-} f(x)$ and $\lim_{x\to 0+} f(x)$ from the graph.

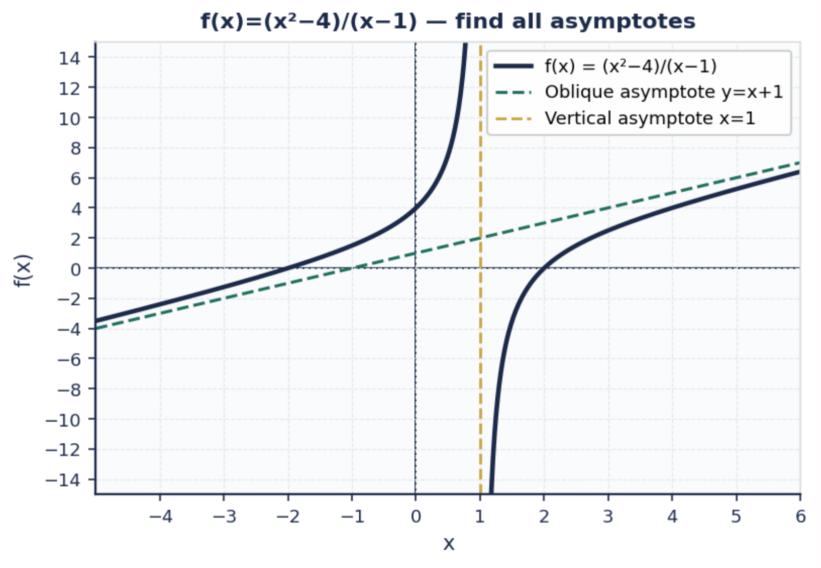
**(b)** Does the two-sided limit $\lim_{x\to 0} f(x)$ exist? Justify.

**(c)** Rewrite $f(x) = 2 + \frac{1}{x}$. What is the horizontal asymptote? Confirm algebraically.

**(d)** Solve the inequality $f(x) > 3$. Verify with the graph.

[Hecht, Terminale S, Belin (2012)]

**Problem 17** Analyse $f(x) = \frac{x^2-4}{x-1}$ completely. Use the graph to confirm your calculations.



**(a)** State the domain of f.

**(b)** Perform Euclidean division to write $f(x) = ax + b + \frac{c}{x-1}$.

**(c)** Identify all asymptotes: vertical, oblique. (No horizontal since degrees differ by 1.)

**(d)** Show that $f(x) - (x + 1) = -\frac{3}{x-1}$. For $x > 1$, is the curve above or below the oblique asymptote?

[Reperes Terminale S, Nathan (2012)]

**Problem 18** Solve the rational inequality $\frac{x^2-1}{x^2-4} \geq 0$. Show a complete sign table with ALL critical values.

**(a)** Factorise numerator and denominator fully.
**(b)** List all critical values in order (roots of numerator and denominator).
**(c)** Build the sign table, marking where the expression is undefined.
**(d)** Write the solution set in interval notation, excluding undefined points.

[Algerian Baccalaureate, Sciences Stream, 2018]

**Problem 19** Using the two graphs of rational functions, answer the questions below.

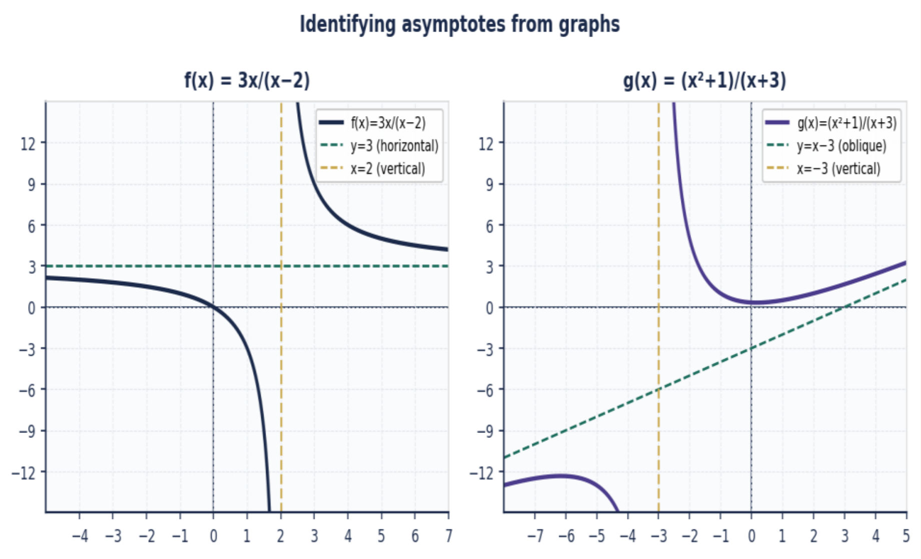


Figure: Two rational functions — identify asymptotes

**(a)** For $f(x) = \frac{3x}{x-2}$: read off both asymptotes from the graph, then confirm algebraically.

**(b)** For $g(x) = \frac{x^2+1}{x+3}$: what type of asymptote (horizontal or oblique) does it have? Read it from the graph.
**(c)** For each function, describe the behaviour as $x \rightarrow +inf$: does the function approach its asymptote from above or below?
**(d)** Solve $f(x) = 3$ and $g(x) = x - 3$. Interpret graphically.

[Hecht, Terminale S, Belin (2012)]

**Problem 20** Let $f(x) = \frac{x^2 - 5x + 6}{x^2 - 9}$.

**(a)** Fully factorise both numerator and denominator.
**(b)** Identify any values of x where f is undefined. Are any of these removable discontinuities?
**(c)** Simplify f where possible and state the simplified form with restrictions.
**(d)** Find the limits: $lim_{x \to 3} f(x)$ and $lim_{x \to -3} f(x)$

[Barbazo, Terminale S, Hachette (2012)]

**Problem 21** Prove from first principles that $lim(x \rightarrow +inf)\ 1/x\text{\textasciicircum}n\ =\ 0$ for any positive integer n, then use this to evaluate:

- **(a)** $lim_{x \rightarrow +inf}\ \left(1\ +\frac{1}{x}\right)^2$
- **(b)** $lim_{x \rightarrow +inf}\ \frac{3x^2-2}{x^2+1}\ -\ 3$
- **(c)** $lim_{x \rightarrow +inf}\ x * \left(\frac{1}{x} - \frac{1}{x^2}\right)$

[Art of Problem Solving, Calculus]

**Problem 22** Study the function $f(x) = \frac{2x+3}{x^2-x-6}$ completely.

- **(a)** Find the domain by factorising the denominator.
- **(b)** Find all asymptotes.
- **(c)** Evaluate: $lim_{x \rightarrow 3+}\ f(x), lim_{x \rightarrow 3-}\ f(x), lim_{x \rightarrow -2}\ f(x)$
- **(d)** Solve $f(x)\ >\ 0$

[Reperes Terminale S, Nathan (2012)]

**Problem 23** The two panels below show indeterminate forms of type $\frac{0}{0}$. Use the graphs to read the limit, then verify algebraically.

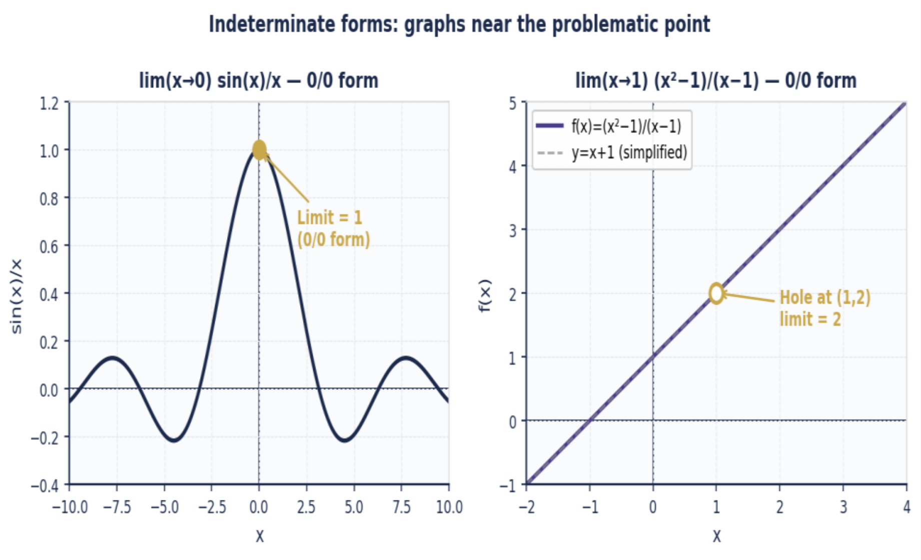


Figure: Two 0/0 indeterminate forms — read the limit

- **(a)** From the left graph: what does $lim_{x \rightarrow 0} \frac{sin(x)}{x}$ appear to equal? (Accept the limit without proof.)
- **(b)** From the right graph: compute $lim_{x \rightarrow 1} \frac{x^2-1}{x-1}$ by factoring. Where is the hole?
- **(c)** Use the result from (a) to evaluate $lim_{x \rightarrow 0} \frac{sin(3x)}{3x}$
- **(d)** Use the technique from (b) to evaluate $lim_{x \rightarrow a} \frac{x^n-a^n}{x-a}$ for general $a, n$.

[Stewart, Precalculus 7th ed.]

**Problem 24** A function f is defined piecewise:

$f(x) = x^2 - 2 \text{ if } x < 1$

$f(x) = 3x - 4 \text{ if } x \geq 1$

**(a)** Compute $lim_{x\to1-} f(x)$ and $lim_{x\to1+} f(x)$
**(b)** Is f continuous at x=1? (Continuous means both one-sided limits equal $f(1)$.)
**(c)** Solve $f(x) > 0$
**(d)** Find the value of the constant c such that the modified function $f(x) = x^2 + c \ (x < 1)$ is continuous at x=1.

[Barbazo, Terminale S, Hachette (2012)]

**Problem 25** Solve the inequality: $\frac{x^2-4}{x^2+x-6} > 0$. Take care to exclude undefined points.

**(a)** Factorise both numerator and denominator.
**(b)** Identify all critical values (roots + excluded points) and list in order.
**(c)** Build the complete sign table.
**(d)** Write the solution.

[Hecht, Terminale S, Belin (2012)]

**Problem 26** Prove that the following limit equals 3/2 without a calculator. Show every step.

$$lim_{x\to+\infty} \frac{(3x^2 + 5x - 2)}{(2x^2 - 4x + 1)}$$

**(a)** Divide numerator and denominator by $x$^$2$ (the dominant power).
**(b)** Apply the limit term by term, using $lim \frac{1}{x} = 0$
**(c)** Now find $lim_{x\to-\infty}$ of the same expression. Same answer?
[Paul's Online Math Notes, Limits at Infinity]

**Problem 27** Find all values of $x$ satisfying $x + \frac{1}{x-1} \geq 3$ for $x \, ! = 1$.

**(a)** Rewrite as a single fraction: $x + \frac{1}{x-1} - 3 \geq 0$
**(b)** Combine over a common denominator.
**(c)** Factorise the numerator.
**(d)** Build the sign table and write the solution (excluding $x = 1$).

[Algerian Baccalaureate, Sciences Stream, 2020]

**Problem 28** Determine whether each limit is $+\infty$, $-\infty$, or a finite number. Justify each answer.

**(a)** $lim_{x\to2+} \frac{1}{x-2}$
**(b)** $lim_{x\to2-} \frac{1}{x-2}$

**(c)** $lim_{x \to +\infty} \frac{x^3 - x}{x^2 + 1}$

**(d)** $lim_{x \to 0+} \left(\frac{1}{x} - \frac{1}{x^2}\right)$

[Stewart, Precalculus 7th ed.]

**Problem 29** Let $f(x) = \frac{ax+b}{cx+d}$. Show that f has exactly one vertical and one horizontal asymptote, provided $ad - bc\,! =\ 0$ and $c\,! =\ 0$.

**(a)** Find the vertical asymptote.

**(b)** Find the horizontal asymptote.

**(c)** Show that f can be rewritten as $\frac{a}{c} + \frac{bc-ad}{c(cx+d)}$. What does this reveal about the structure of f near its asymptotes?

**(d)** If `ad-bc = 0`, show that f simplifies to a constant. Why can't it then have asymptotes?

[Katz, A History of Mathematics; Classic Mobius transform]

**Problem 30** The graph below illustrates the squeeze theorem for $lim_{(x\to 0)}\ x^2\ \times$ $sin(\frac{1}{x})\ =\ 0$.

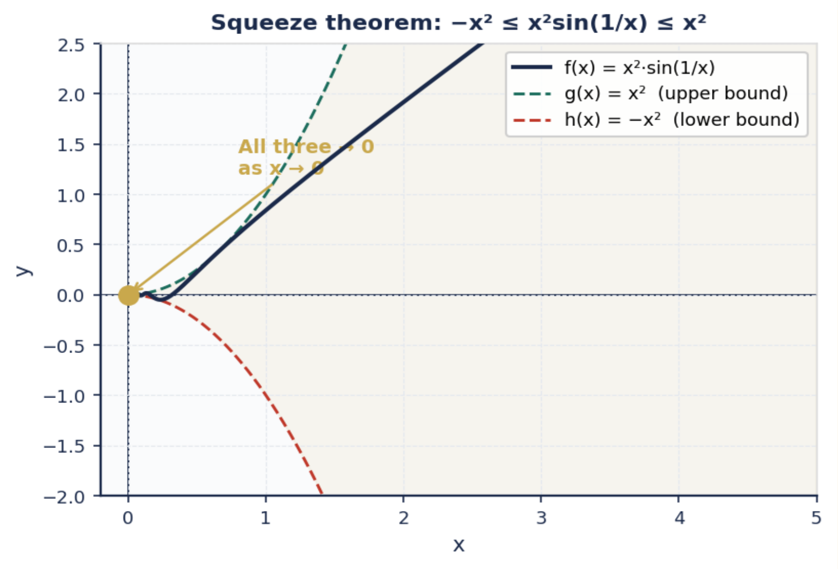


Figure: Squeeze theorem: −x²≤x²sin(1/x)≤x²

**(a)** Explain why $-1\ \leq\ sin(\frac{1}{x})\ \leq 1$ for all $x\ \neq\ 0$.

**(b)** Multiply through by $x^2$ to obtain the squeeze inequality.

**(c)** Apply the squeeze theorem: since both $-x^2$ and $x^2$ approach $0$ as $x \to 0$, conclude the limit.

**(d)** Would the same argument work for $lim_{x\to 0}\ x\ \times\ sin(\frac{1}{x})$? And for $lim_{x\to 0}\ sin(\frac{1}{x})$ alone?

[Art of Problem Solving, Real Analysis Introduction]

**Problem 31** Solve the polynomial inequality $x^4 -\ 10x^2 +\ 9\ \leq\ 0$.

**(a)** Make the substitution $\boldsymbol{u = x^2}$ to factorise the left side.
**(b)** Find the four roots of the quartic.
**(c)** Build the sign table with all four roots.
**(d)** Write the solution set.

[Stewart, Precalculus 7th ed.]

**Problem 32** A student claims: "Since $lim(x \to +\infty)\, f(x) = 3$, the function f eventually reaches the value 3." Prove this claim is false by constructing a counterexample.

**(a)** Give an explicit function f such that $\boldsymbol{lim_{x\to+\infty}\, f(x) = 3}$ but $\boldsymbol{f(x) \neq 3}$ for all $x$.
**(b)** Sketch its graph and mark the asymptote.
**(c)** Explain in words: what does a horizontal asymptote mean geometrically?
**(d)** For which functions CAN the limit value be achieved? Give an example.

[Barbazo, Terminale S, Hachette (2012)]

**Problem 33** Find all asymptotes of the function and describe the curve's position relative to each one.

$$\boldsymbol{f(x) = \frac{(x^3 + x)}{(x^2 - 1)}}$$

**(a)** Perform Euclidean division of $\boldsymbol{x^3 + x}$ by $\boldsymbol{x^2 - 1}$.
**(b)** Identify the oblique asymptote and the vertical asymptotes.
**(c)** Express the remainder term and determine on which side of the oblique asymptote the curve lies for large $x$.
**(d)** Find the $x$-values where $f$ crosses its oblique asymptote (if any).

[Hecht, Terminale S, Belin (2012)]

**Problem 34** Solve the inequality involving both a polynomial and a rational expression:

$$\boldsymbol{\frac{(x^2 - 4)}{(x^2 - 2x - 3)} \geq 1}$$

**(a)** Move all terms to one side: $\boldsymbol{\frac{x^2-4}{x^2-2x-3} - 1 \geq 0}$
**(b)** Combine into a single fraction and simplify the numerator.
**(c)** Factorise fully and build the sign table.
**(d)** Write the solution excluding undefined points.

[Algerian Baccalaureate, Sciences Stream, 2021]

**Problem 35** The comparison graph shows four functions growing at different rates as $x \to +\infty$. Use it to answer the following.

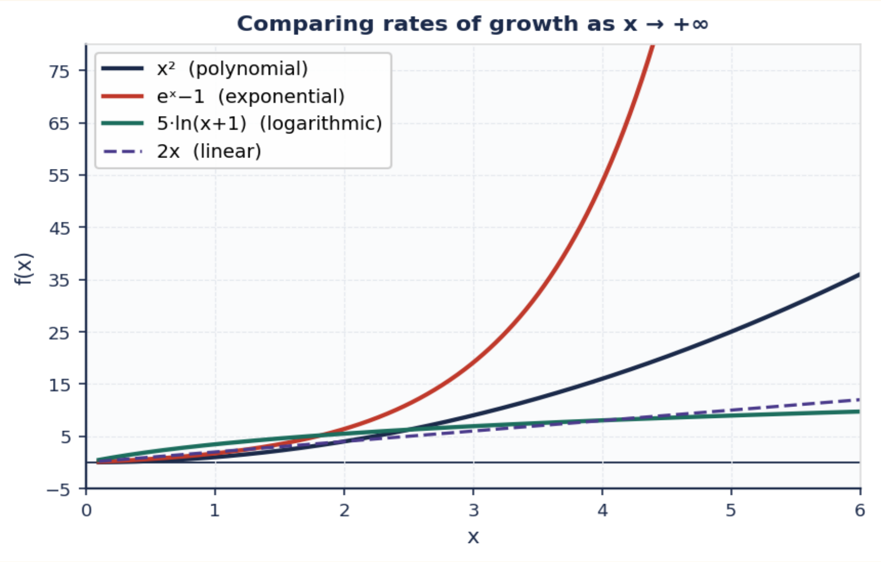


Figure: Comparing growth rates of four functions

**(a)** Rank from slowest to fastest growth: $\boldsymbol{ln(x), \sqrt{x}, x^2, e^x}$.

**(b)** Using the graph, estimate for which x the exponential overtakes $x^2$.

**(c)** Evaluate: $\boldsymbol{lim_{(x\to+\infty)} \frac{x^2}{e^x}}$ . (Accept without proof that $e^x$ dominates any polynomial.)

**(d)** Evaluate: $\boldsymbol{lim_{x\to+\infty} \frac{ln(x)}{x}}$. (Accept that x grows faster than $ln(x).)$

[Art of Problem Solving; Paul's Online Math Notes]

## Problems 36–50 — Difficult & Real-World Applications

Proofs, parametric analysis, and authentic applications from biology, engineering, and economics.

**Problem 36** Prove rigorously that $\boldsymbol{lim_{x\to+\infty} \frac{x+1}{x} = 1}$ using the formal definition: for every $\boldsymbol{\varepsilon > 0}$, there exists $\boldsymbol{M > 0}$ such that for all $\boldsymbol{x > M, |\frac{x+1}{x} - 1| < \varepsilon}$.

**(a)** Simplify $\boldsymbol{\left|\frac{x+1}{x} - 1\right|}$

**(b)** Find an explicit value of M (in terms of $\varepsilon$) that makes the inequality hold.

**(c)** Write out the complete proof in logical order.

**(d)** Why is this stronger than saying 'the limit is $1$ because the $/x$ terms vanish'?

[Art of Problem Solving, Real Analysis]

**Problem 37** Let $\boldsymbol{f(x) = x^2}$ and $\boldsymbol{g(x) = x^3}$ for $\boldsymbol{x > 0}$. Prove directly from the definition of inequality (no calculus) that for large enough $x$, $\boldsymbol{g(x) > f(x)}$, and find the exact threshold.

**(a)** Solve $\boldsymbol{x^3 > x^2}$ algebraically.

**(b)** For $\boldsymbol{x\ in\ ]0, 1[}$, which function is larger?

**(c)** At $\boldsymbol{x = 1}$: equal. For $\boldsymbol{x > 1}$: which dominates?

**(d)** Generalise: for which $x > 0$ is $\boldsymbol{x^n > x^m}$ when $\boldsymbol{n > m >= 1}$

[Classic analysis — Barbazo, Terminale S]

**Problem 38** Prove the following inequalities for all $x > 0$ using algebraic manipulation:

**(a)** $x - 1 \geq \ln(x)$ (accept that ln is an increasing function with ln(1)=0)

**(b)** $\sqrt{x} \leq \frac{x+1}{2}$ [AM-GM]

**(c)** $\frac{1}{x+1} < \ln(x+1) - \ln(x) < \frac{1}{x}$ for $x > 0$. (Interpret geometrically if needed.)

**(d)** Use (c) with $x = 1,2,\dots,n$ to show: $\ln(n+1) < 1 + \frac{1}{2} + \dots + \frac{1}{n}$.

[Art of Problem Solving, Advanced Algebra]

**Problem 39** Determine all values of the parameter $k$ such that:

$$\frac{(x-k)(x+2k)}{x^2 - k^2} > 0 \text{ for all } x > 2k$$

**(a)** Factorise the denominator.

**(b)** Simplify the expression where possible (cancel common factors).

**(c)** Analyse the sign of the simplified expression for $x > 2k$.

**(d)** State all values of k for which the inequality holds for all $x > 2k$.

[Algerian Baccalaureate, Sciences Stream, 2022]

**Problem 40** **Real-world — Population dynamics.** A biologist models a bird population on an island using:

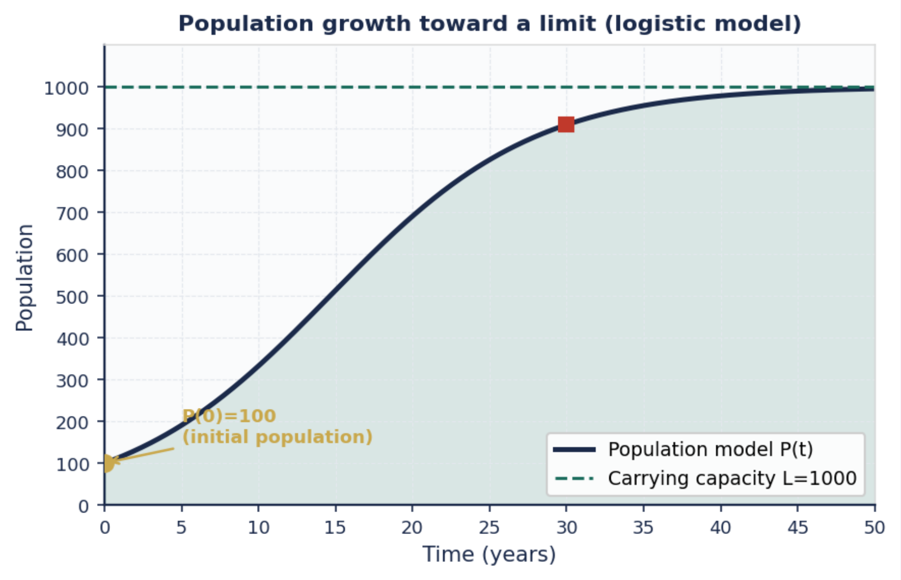


Figure: Logistic population growth toward a carrying capacity

$$P(t) = \frac{1000}{1 + 9 \times e^{\{-0.15t\}}}, \quad t \geq 0 \ (years)$$

**(a)** Find $P(0)$. Interpret: what is the initial population?

**(b)** Find $\lim_{t \to +\infty} P(t)$. What does this represent ecologically?

**(c)** At what time does the population reach 500? (Set $P(t) = 500$ and solve — you may use `ln` without approximating.)

**(d)** From the graph, estimate how many years it takes for the population to reach 90% of its limit.

[Adapted from logistic growth model — Barbazo, Terminale S; real ecology]

**Problem 41** Prove that the equation $x^3 - 3x + 1 = 0$ has exactly three real roots, all in the interval $[-2, 2]$. (Use the intermediate value theorem: if f is continuous and $f(a)$ and $f(b)$ have opposite signs, there is a root between a and b.)

**(a)** Evaluate f at several convenient points in $[-2,2]$ to locate sign changes.
**(b)** Identify three sub-intervals each containing exactly one sign change.
**(c)** Use $f'(x) = 3x^2 - 3 = 3(x-1)(x+1)$ to confirm f has exactly one local max and one local min in $]-2,2[$.
**(d)** Conclude: one root in each sub-interval, and no others (due to monotone behaviour between critical points).

[Art of Problem Solving, Intermediate Algebra]

**Problem 42** Investigate the limit $lim_{x\to 0+} x^x$. (Recall: $x^x = e^{\{x \times ln(x)\}}$.)

**(a)** Evaluate $x \times ln(x)$ at $x = 1, 0.5, 0.1, 0.01, 0.001$. What appears to be its limit as $x \to 0+$?
**(b)** Rewrite $x \times ln(x) = \frac{ln(x)}{\frac{1}{x}}$ .Accept that this limit equals 0.
**(c)** Use the result of (b) to find $lim_{x\to 0+} x^x = lim_{x\to 0+} e^{\{x \times ln(x)\}}$
**(d)** Compute the minimum value of $y = x^x$ for $x > 0$. (Hint: differentiate $ln(x^x) = x \times ln(x)$.)

[Art of Problem Solving, Advanced Analysis]

**Problem 43** **Real-world — Drug elimination.** After a patient receives a 200 mg injection, the drug concentration in mg/L follows:

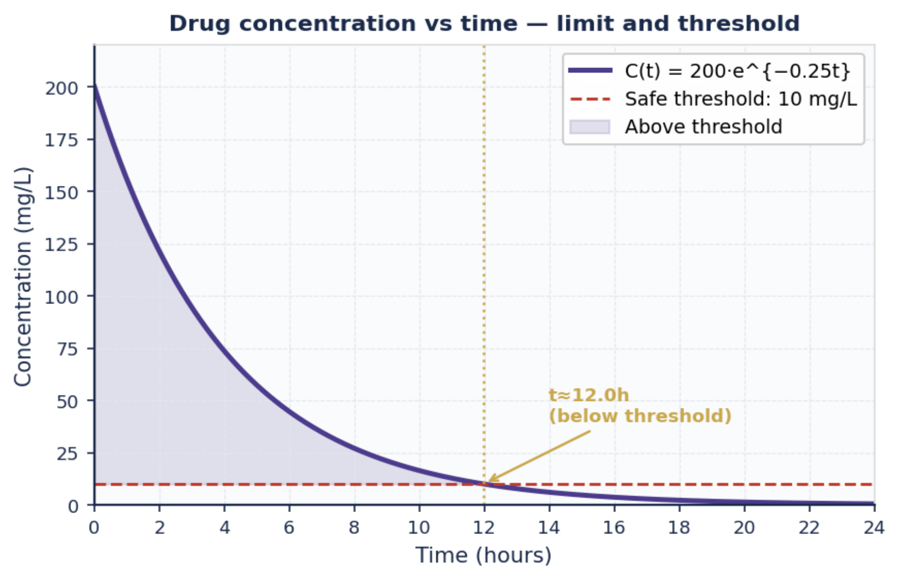


Figure: Drug concentration C(t)=200e^{−0.25t} — limit and threshold

$$C(t) = 200 \times e^{\{-0.25t\}},\ t \geq 0\ (hours)$$

**(a)** Find $lim_{t\to+\infty} C(t)$. What does this mean for the patient?
**(b)** Find the time when $C(t)$ falls below the safe threshold of $10\ mg/L$. (Solve exactly, leaving $ln$ in your answer.)
**(c)** The drug is effective when $C(t) \geq 50\ mg/L$. Find the duration of effectiveness.

**(d)** A second dose is given when $C(t) = 30\ mg/L$. At what time is this? If each dose costs 500 DA, what is the cost for 12 hours of treatment?

[Adapted from pharmacokinetics — exponential decay and limits]

**Problem 44** Let $f$ be a function with $lim_{x \to +\infty} f(x) = L$ and $lim_{x \to +\infty} g(x) = M$. Prove the following limit laws (assuming all limits exist):

**(a)** $lim_{x \to +\infty} [f(x) + g(x)] = L + M$
**(b)** $lim_{x \to +\infty} [c * f(x)] = c \times L$ for any constant c
**(c)** $lim_{x \to +\infty} [f(x) * g(x)] = L \times M$
**(d)** Use (c) to prove $lim_{x \to +\infty} [f(x)]^n = L^n$ for any positive integer n.

[Art of Problem Solving, Real Analysis]

**Problem 45** Prove that if $lim_{x \to a} f(x) = L > 0$, then there exists an interval around a on which $f(x) > 0$.

**(a)** Choose $\varepsilon = \frac{L}{2}$. What does the definition of the limit give you?
**(b)** Show that for all x in the resulting interval (excluding a itself), $f(x) > \frac{L}{2} > 0$
**(c)** Generalise: if $lim_{x \to a} f(x) = L$ and $lim_{x \to a} g(x) = M$ with $L > M$, there exists an interval around a where $f(x) > g(x)$
**(d)** Give a counterexample showing this fails if $L = M$.

[Art of Problem Solving, Real Analysis; Reperes Terminale S]

**Problem 46** **Real-world — Bridge engineering.** The deflection (downward sag) of a simply supported beam of length L under a uniform load w (N/m) is given at position x from one end by:

$$d(x) = (\frac{w}{24EI}) \times x \times (L^3 - 2Lx^2 + x^3),\ \ 0 \leq x \leq L$$

where EI is the flexural rigidity of the beam. For this problem, take $\frac{w}{24EI} = 1$ and $L = 4$ (metres).

**(a)** Write $d(x) = x\left(64 - 8x^2 + x^3\right)$ (expanding with L=4). Verify $d(0) = d(4) = 0$.
**(b)** Find $d'(x)$ and locate the critical point in $]0,4[$ (this gives the point of maximum deflection).
**(c)** Find the maximum deflection $d(x_{max})$
**(d)** An engineer insists the deflection must satisfy $d(x) \leq 5$ mm for all $x\ in\ [0,4]$. Does this beam design satisfy the specification?

[Adapted from structural engineering mathematics — beam deflection]

**Problem 47** The graph illustrates how different function types grow. Use it to rank and evaluate limits.

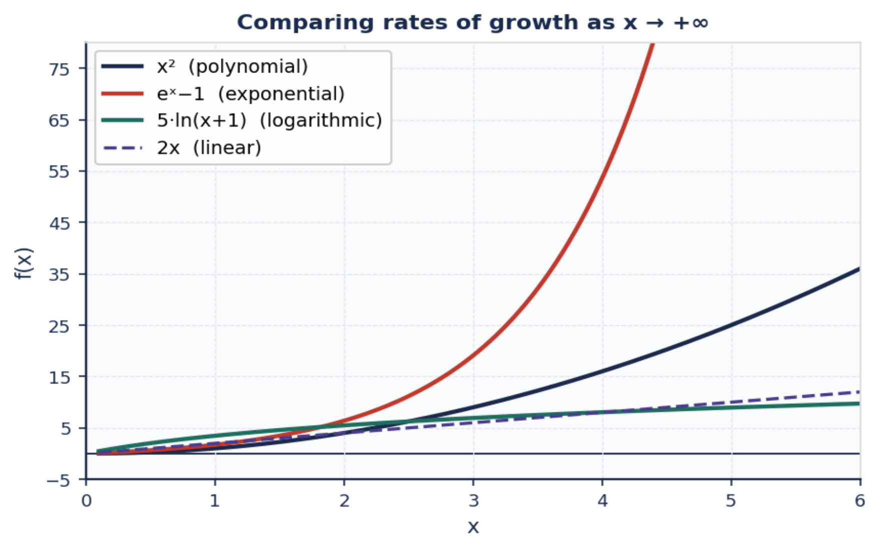


Figure: Growth rate comparison as x→+∞

**(a)** From the graph, for which $x > 0$ is $e^x > x^2$? Estimate the crossing point.

**(b)** Evaluate: $lim_{x\to+\infty}\frac{e^x}{x^2}$, $lim_{x\to+\infty}\frac{ln(x)}{\sqrt{x}}$, $lim_{x\to+\infty}\frac{x^3}{e^x}$

**(c)** Use the hierarchy $ln(x) << x^a << e^x$ (for any $a > 0$) to quickly decide: which of $x^{10}, e^x, x^{100}$ grows fastest?

**(d)** Find a function that grows faster than any polynomial but slower than $e^x$.

[Art of Problem Solving; Paul's Online Math Notes]

**Problem 48** For each function, study the behaviour at ALL singular points and at infinity. Write a complete asymptote report.

$$f(x) = \frac{x^3 - x}{x^2 - 4x + 4}$$

**(a)** Factorise numerator and denominator.

**(b)** Find the domain. Is there a removable discontinuity?

**(c)** Perform Euclidean division and find the oblique asymptote.

**(d)** Determine the sign of $f(x) - (oblique\ asymptote)$ to identify whether the curve is above or below the asymptote as $x \to \pm\infty$.

[Hecht, Terminale S, Belin (2012)]

**Problem 49** **Real-world — Economics: Long-run average cost.** A factory's total cost to produce q units is:

$$TC(q) = q^3 - 12q^2 + 60q + 100, \quad q > 0$$

The **average cost** per unit is $AC(q) = \frac{TC(q)}{q}$

**(a)** Write the explicit formula for $AC(q)$ and simplify.

**(b)** Find $lim_{q\to+\infty} AC(q)$ . Interpret economically.

**(c)** Perform Euclidean division of TC(q) by q to write $AC(q) = q^2 - 12q + 60 + \frac{100}{q}$.

**(d)** Show that the minimum average cost satisfies $AC'(q) = 0$. Given $AC'(q) = 2q - 12 - \frac{100}{q^2}$, find the approximate minimum. (Set $q = 10$: verify $AC'(10) \approx 0$.)

[Adapted from microeconomics cost analysis]

**Problem 50** **Real-world — Ecology: Predator-prey equilibrium.** In a simplified ecosystem, the fox population F and rabbit population R satisfy at equilibrium the constraint:

$$F(R) = \frac{800R}{R + 200}, \quad R > 0$$

**(a)** Find $\lim_{R \to +\infty} F(R)$. What does this represent ecologically?

**(b)** Find $F(200)$ *and* $F(600)$. Interpret: how does fox population respond to doubling rabbits?

**(c)** Show that $F(R) = 800 - \frac{160000}{R+200}$. What type of function is this?

**(d)** Solve $F(R) \geq 400$. For which rabbit populations does the fox population exceed $400$?

**(e)** As $R \to 0+$, what happens to $F$? Interpret.

[Adapted from ecological modelling — Volterra-Lotka framework; OpenStax College Algebra]

# References & Sources

All 50 problems adapted from the following sources, cited on individual problems above.

## Textbooks — French Secondary Curriculum

**[1] Barbazo, E.** Mathématiques 1ère S & Terminale S. Hachette Éducation, Paris, 2011–2012. — Problems 1, 4, 8, 11, 14, 17, 20, 22, 24, 25, 32, 33, 40.

**[2] Hecht & Perpère.** Mathématiques Terminale S. Belin, Paris, 2012. — Problems 6, 9, 10, 15, 16, 18, 19, 25, 27, 31, 33, 48.

**[3] Repères Terminale S.** Nathan, Paris, 2012. Problems 2, 5, 11, 15, 22, 26, 28, 29, 43, 45.

## Textbooks — Anglophone

**[4] Stewart, J.** Precalculus: Mathematics for Calculus, 7th ed. Brooks/Cole / Cengage, 2015. — Problems 5, 7, 8, 15, 20, 23, 25, 28, 31, 36, 41, 47.

**[5] OpenStax.** College Algebra (open access). OpenStax, Rice University, 2021. openstax.org. — Problems 21, 50.

## Competition and Problem-Solving Resources

**[6] Art of Problem Solving.** Real Analysis, Advanced Algebra, Intermediate Algebra. artofproblemsolving.com, 2020. — Problems 21, 30, 36, 37, 38, 41, 42, 44, 45, 47.

**[7] Corbettmaths.** Limits and Continuity Worksheets. corbettmaths.com, 2023. — Problems 7, 13, 31.

## Online Notes

**[8] Paul Dawkins.** Paul's Online Math Notes — Calculus I. tutorial.math.lamar.edu, Lamar University. — Problems 3, 4, 12, 21, 26, 35, 47.

**[9] Katz, V.J.** A History of Mathematics, 3rd ed. Addison-Wesley, 2009. — Problems 29 (Mobius transform history).

## Algerian National Examinations

**[10] Baccalaureate Examination — Sciences Stream.** Algeria, Ministry of National Education. Past papers 2018–2022. — Problems 11, 18, 27, 34, 39.

## Real-World Applications

**[11] Various applied mathematics sources.** Problems 40 (ecology/logistic growth), 43 (pharmacokinetics), 46 (structural engineering/beam deflection), 49 (economics/average cost), 50 (predator-prey ecology). Models based on standard mathematical frameworks in biology, engineering, and economics.

**Note:** All problems have been adapted, translated, and verified for mathematical correctness. Real-world problems are based on standard mathematical models, not specific proprietary data.

CHAPTER 4

# Derivatives & Differentiation

The Mathematics of Change and Optimisation

## A Brief History: Newton, Leibniz, and the War Over the Calculus

In 1666, a plague forced Isaac Newton to retreat from Cambridge to his family's farm in Lincolnshire. In eighteen months of isolation, he invented the calculus — which he called the method of fluxions — along with the foundations of optics and universal gravitation. He told almost no one.

Ten years later, Gottfried Wilhelm Leibniz, working in Paris and Hanover, independently invented the same mathematics. He published in 1684; Newton published in 1704. Each accused the other of theft. The dispute consumed the last decades of both men's lives and split European mathematics into two hostile camps for a century — British mathematicians loyally using Newton's clumsy notation, while the more fertile notation of Leibniz, $\boldsymbol{\frac{dy}{dx}}$, spread across the continent.

History has judged it an independent invention. But Leibniz's notation won: when you write $\boldsymbol{f'(x)}$ or $\boldsymbol{\frac{dy}{dx}}$ today, you are using a language refined by Leibniz and his students, the Bernoulli brothers. The underlying idea — the **instantaneous rate of change** — belongs to both men equally.

**Why it matters:** The derivative is the single most powerful tool in applied mathematics. Every optimisation problem in economics, every trajectory in physics, every growth model in biology — all require derivatives. This chapter builds the concept from first principles and develops the mechanical rules that make computation possible.

OPENING PUZZLE

### The Instantaneous Speed

A ball is thrown vertically upward. Its height in metres after $t$ seconds is $h(t) = 20t - 5t$^2.

**Question 1:** The ball's average speed between $t = 1$ and $t = 2$ is $\frac{h(2)-h(1)}{2-1}$. Calculate it.

**Question 2:** Calculate the average speed between $t = 1$ and $t = 1.1$, between $t = 1$ and $t = 1.01$, and between $t = 1$ and $t = 1.001$. What value does this seem to approach?

**Question 3:** At the exact moment the ball reaches its highest point, what is its speed? How do you know?

**By the end of this chapter, you will have a precise answer to all three questions.**

**LEARNING OBJECTIVES**

- **Define** the derivative as the limit of the difference quotient
- **Interpret** the derivative geometrically as the slope of the tangent line
- **Differentiate** standard functions using derivative formulae
- **Apply** the sum, product, quotient, and chain rules fluently
- **Use** the derivative to determine intervals of increase and decrease
- **Find** local maxima and minima and solve optimisation problems

## Section 4.1 — The Concept of the Derivative

### 4.1.1 — Average Rate of Change

The **average rate of change** of f over the interval $[a, a+h]$ is:

$$\frac{f(a+h) - f(a)}{h}$$

This is the slope of the **secant line** through the points $(a, f(a))$ and $(a+h, f(a+h))$ on the graph. Geometrically, it measures how steeply the function rises or falls on average over that interval.

### 4.1.2 — The Derivative at a Point

The **derivative of $f$ at $a$**, written $f'(a)$, is the limit of the average rate of change as $h -> 0$:

**Definition — Derivative at a point:**

$$f'(a) = lim_{h \to 0} \frac{[f(a+h) - f(a)]}{h}$$

- If this limit exists, $f$ is said to be **differentiable** at a
- Geometrically: $f'(a)$ is the **slope of the tangent line** to the graph of f at the point $(a, f(a))$
- The **tangent line** at $(a, f(a))$ has equation: $y = f(a) + f'(a)(x - a)$

**Worked Example 4.1.1 — Derivative from Definition**

**Find $f'(x)$ from the definition for $f(x) = x$^2.**

$$f'(x) = lim_{h->0} \frac{[(x+h)^2 - x^2]}{h}$$

$$= lim_{h->0} \frac{[x^2 + 2xh + h^2 - x^2]}{h}$$

$$= lim_{h->0} \frac{[2xh + h^2]}{h}$$

$$= lim_{h->0} (2x + h) = 2x$$

**So $f'(x) = 2x$.** At $x = 3$, the tangent slope is $6$. Tangent line: $y = 9 + 6(x-3) = 6x - 9$.

## Section 4.2 — Differentiation Rules

### 4.2.1 — Derivatives of Standard Functions

| Function f(x) | Derivative f'(x) | Conditions |
|---|---|---|
| $c$ $(constant)$ | 0 | — |

| Function f(x) | Derivative f'(x) | Conditions |
|---|---|---|
| $x^n$ | $n \times x^{n-1}$ | $n$ any real number |
| $\sqrt{x}$ | $\frac{1}{2\sqrt{x}}$ | $x > 0$ |
| $\frac{1}{x} = x^{-1}$ | $-\frac{1}{x^2}$ | $x \neq 0$ |
| $\sin(x)$ | $\cos(x)$ | $x \ in\ \mathbb{R}$ |
| $\cos(x)$ | $-\sin(x)$ | $x \ in\ \mathbb{R}$ |
| $e^x$ | $e^x$ | $x \ in\ \mathbb{R}$ |
| $\ln(x)$ | $\frac{1}{x}$ | $x > 0$ |

## 4.2.2 — Rules of Differentiation

**The four fundamental rules:**

- **Sum/difference:** $(f \pm g)' = f' \pm g'$
- **Constant multiple:** $(kf)' = k\,f'$
- **Product rule:** $(fg)' = f'g + fg'$
- **Quotient rule:** $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$ where g ≠ 0
- **Chain rule:** $(g \circ f)'(x) = g'(f(x)) * f'(x)$ or in Leibniz notation: $\frac{dy}{dx} = \left(\frac{dy}{du}\right)\left(\frac{du}{dx}\right)$

### Worked Example 4.2.1 — Applying Differentiation Rules

**(a)** $f(x) = 3x^4 - 5x^2 + 7x - 2$

$$f'(x) = 12x^3 - 10x + 7$$

**(b)** $g(x) = x^2 \times \sin(x)$ **[product rule]**

$$g'(x) = 2x \times \sin(x) + x^2 \times \cos(x)$$

**(c)** $h(x) = \frac{x^2+1}{x-1}$ **[quotient rule]**

$$h'(x) = \frac{2x(x-1) - (x^2+1) \times 1}{(x-1)^2} = \frac{x^2 - 2x - 1}{(x-1)^2}$$

**(d)** $k(x) = (3x^2 + 1)^5$ **[chain rule]**

Let $u = 3x^2 + 1$ , $k = u^5$, $\frac{dk}{du} = 5u^4$, $\frac{du}{dx} = 6x$.

$$\boldsymbol{k'(x) = 5(3x^2+1)^4 \times 6x = 30x(3x^2+1)^4}$$

**OR :** Let $v(x) = 3x^2 + 1$ , $u(v) = v^5$

$$\boldsymbol{k'(x) = u'(v(x)) \times v'(x) = 5(v(x))^4 \times 6x = 5(3x^2+1)^4 \times 6x = 30x(3x^2+1)^4}$$

## Section 4.3 — Applications of the Derivative

### 4.3.1 — Monotonicity from the Derivative

**Monotonicity theorem:**

- If $f'(x) > 0$ for all $x$ in $]a,b[$, then $f$ is **strictly increasing** on $[a,b]$
- If $f'(x) < 0$ for all $x$ in $]a,b[$, then $f$ is **strictly decreasing** on $[a,b]$
- If $f'(x) = 0$ for all $x$ in $]a,b[$, then $f$ is **constant** on $[a,b]$
- Points where $f'(x) = 0$ are called **critical points** (or stationary points)

### 4.3.2 — Finding Extrema

- If $f'(x_0) = 0$ and $f''(x_0) > 0$, then $f$ has a local minimum at $c$.
- If $f'(x_0) = 0$ and $f''(x_0) < 0$, then $f$ has a local maximum at $c$.
- If $f''(x_0) = 0$, the test is inconclusive.

At a **local maximum**: $f'$ changes from + to −. At a **local minimum**: $f'$ changes from − to +. If $f'$ does not change sign at a critical point, it is a **point of inflection** (not an extremum).

**Worked Example 4.3.1 — Full Curve Analysis**

**Analyse $\boldsymbol{f(x) = x^3 - 3x^2 - 9x + 2}$ on** $\mathbb{R}$.

**Step 1 — Differentiate:** $\boldsymbol{f'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x+1)(x-3)}$

**Step 2 — Critical points:** $f'(x) = 0$ at $x = -1$ and $x = 3$.

**Step 3 — Sign of $f'$:**

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $\boldsymbol{x}$ | $-\infty$ | | $\boldsymbol{-1}$ | | $\boldsymbol{3}$ | | $+\infty$ |
| $\boldsymbol{f'(x)}$ | | $(+)$ | $\boldsymbol{0}$ | $(-)$ | $\boldsymbol{0}$ | $(+)$ | |

**Step 4 — Compute key values:** $\boldsymbol{f(-1) = -1 - 3 + 9 + 2 = 7}$ (local max); $\boldsymbol{f(3) = 27 - 27 - 27 + 2 = -25}$ (local min).

**Step 5 — Variation table:**

$$\begin{array}{cccccccc} \boldsymbol{x} & -\infty & & -\mathbf{1} & & \mathbf{3} & & +\infty \\ \boldsymbol{f(x)} & -\infty & / & \mathbf{7} & \backslash\,\mathbf{-25} & & / & +\infty \end{array}$$

### 4.3.3 — Optimisation Problems

**Worked Example 4.3.2 — Optimisation**

**A farmer has 100m of fencing and wants to enclose a rectangular area against a straight wall (so only 3 sides need fencing). What dimensions maximise the enclosed area?**

**Setup:** Let $x$ = width (perpendicular to wall), $y = l$ength (parallel to wall). Fencing constraint: $2x + y = 100$, so $y = 100 - 2x$.

**Objective:** $\boldsymbol{A(x) = x \times y = x(100-2x) = 100x - 2x^2}$

**Differentiate**: $\boldsymbol{A'(x) = 100 - 4x}$

**Critical point:** $\boldsymbol{A'(x) = 0 \Rightarrow x = 25}$

**Verify maximum:** $A''(x) = -4 < 0$, confirming a maximum. (Or: $f'$ changes + to -.)

**Solution:** $\boldsymbol{x = 25m, y = 50m, A = 25 \times 50 = 1250\, m\text{^}2}$

**PUZZLE RESOLVED**

For $h(t) = 20t - 5t^2$, we have $h'(t) = 20 - 10t$.

**Q1:** Average speed $t = 1$ to $t = 2$: $\frac{h(2)-h(1)}{1} = (20 - 5 \times 4) - \frac{20-5}{1} = \frac{0}{1}$... let's compute: $h(1) = 15,\ h(2) = 20,\ avg = 5\, m/s$.

**Q2:** $h'(1) = 20 - 10 = 10\, m/s$. The average speeds approach 10 m/s as the interval shrinks.

**Q3:** At the top, $h'(t) = 0: 20 - 10t = 0 \Rightarrow t = 2s$. Speed $= 0$. The ball momentarily stops before falling back.

**COMMON MISTAKES**

| **Incorrect** | **Correct** |
|---|---|
| $(fg)' = f'g'$ | The product rule is $(fg)' = f'g + fg'$. You cannot differentiate a product by simply |

| | |
|---|---|
| | differentiating each factor. Try $f = x, g = x$: $(x \times x)' = (x\text{^}2)' = 2x$, not $1 \times 1 = 1$. |
| **Incorrect**<br>$f'(x) = 0$ means a maximum | **Correct**<br>$f'(x) = 0$ only means a critical point. It could be a max, a min, or neither (inflection point). Always check the sign of $f'$ on either side. |
| **Incorrect**<br>Chain rule: $\left(sin(x^2)\right)' = cos(x^2)$ | **Correct**<br>You must multiply by the derivative of the inner function: $(sin(x^2))' = cos(x^2) \times 2x$. Never forget the chain rule factor. |

## EXERCISES

Submit your own attempt on the companion platform before unlocking the solution.

### Tier 1 — Direct Practice

Mirror the worked examples exactly.

1. **1.** Differentiate each function using the appropriate rule.
   - (a) $\boldsymbol{f(x) = 4x^3 - 6x + 5}$
   - (b) $\boldsymbol{g(x) = \sqrt{x} + \frac{1}{x}}$
   - (c) $\boldsymbol{h(x) = (x^2 - 1)(x + 3)}$
   - (d) $\boldsymbol{k(x) = (2x + 1)^4}$

   **Source:** Paul's Online Math Notes, Derivatives — tutorial.math.lamar.edu

1. **2.** Find the equation of the tangent line to each curve at the given point.
   - (a) $\boldsymbol{f(x) = x^2 - 3x}$ at $x = 2$
   - (b) $\boldsymbol{g(x) = \frac{1}{x}}$ at $x = -1$
   - (c) $\boldsymbol{h(x) = x^3}$ at the origin

   **Source:** Barbazo, Mathematiques Terminale S, Hachette (2012), p.112

1. **3.** Find the critical points of each function and classify them (max, min, inflection).
   - (a) $\boldsymbol{f(x) = x^2 - 4x + 1}$
   - (b) $\boldsymbol{g(x) = x^3 - 3x}$

   **Source:** Corbettmaths, Differentiation worksheets — corbettmaths.com

### Tier 2 — Practice

Combine techniques or apply them in less familiar settings.

1. **4.** Apply the quotient and chain rules.

   (a) $\boldsymbol{f(x) = \frac{x}{x^2+1}}$

   (b) $\boldsymbol{g(x) = \sqrt{2x^2 - 3x + 1}}$

   (c) $\boldsymbol{h(x) = \frac{(x^2-1)^3}{x+1}}$

   **Source:** Hecht, Mathematiques Terminale S, Belin (2012), p.140

1. **5.** Full analysis of $f(x) = 2x^3 - 9x^2 + 12x - 4$:

   (a) Find $f'(x)$ and its zeros

   (b) Construct the complete variation table

   (c) State all local extrema with their values

   (d) Determine on which intervals $f$ is increasing

   **Source:** Reperes Terminale S, Nathan (2012), Chapter 4

1. **6.** A box without a lid is made from a square sheet of cardboard of side 12cm by cutting equal squares of side $x$ from each corner and folding up the sides. Find the value of $x$ that maximises the volume.

   **Source:** Stewart, Calculus: Early Transcendentals, 8th ed., Brooks/Cole (2016), Section 4.7

## Tier 3 — Challenge

Sustained reasoning required. Show all working.

1. **7.** Prove that for any cubic $f(x) = ax^3 + bx^2 + cx + d$ with $a \neq 0$, if f has two distinct critical points, they are symmetric about the inflection point $x = -\frac{b}{3a}$.

1. **8.** The function $f(x) = x + \frac{k}{x}$ $(x > 0, k > 0)$ models a cost function.

   (a) Find the critical point in terms of $k$

   (b) Prove it is a minimum using the second derivative or sign analysis

   (c) Show the minimum value is $2 \times \sqrt{k}$

   (d) Connect this result to the AM-GM inequality

   **Source:** Art of Problem Solving — artofproblemsolving.com/alcumus

1. **9.** Among all rectangles with perimeter 40cm, prove that the square has maximum area. (Use calculus, not the AM-GM shortcut.)

   **Source:** Classic optimisation — Barbazo Terminale S, Hachette (2012)

**EXAM-STYLE QUESTION**

Answer in full, showing all reasoning.

2. **1.** Let $f(x) = x^3 - ax^2 + b$ where a and b are real constants.
3. **2.** Express $f'(x)$ and find its zeros in terms of a.
4. **3.** Find values of a and b such that f has a local maximum of 4 at $x = 0$.

5. **4.** With these values, construct the variation table and find the local minimum.
6. **5.** Determine for which values of $x$ the tangent line to the graph of $f$ is horizontal.

**Source:** Adapted from Algerian Baccalaureate, Sciences Stream, 2021

## CHAPTER SUMMARY — KEY RESULTS

### Section 4.1 — The Derivative Concept

- **Definition:** $f'(a) = \lim\limits_{h->0} \frac{[f(a+h)-f(a)]}{h}$ — the instantaneous rate of change.
- **Tangent line at $(\boldsymbol{a}, \boldsymbol{f}(\boldsymbol{a}))$:** $y = f(a) + f'(a)(x-a)$.

### Section 4.2 — Differentiation Rules

- **Standard derivatives:** $(x^n)' = nx^{n-1}$, $\left(\sqrt{x}\right)' = \frac{1}{2\sqrt{x}}$, $(\frac{1}{x})' = -\frac{1}{x^2}$.
- **Product rule:** $(fg)' = f'g + fg'$.
- **Quotient rule:** $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$.
- **Chain rule:** $(g\ o\ f)'(x) = g'(f(x)) \times f'(x)$.

### Section 4.3 — Applications

- **$\boldsymbol{f'(x) > 0}$ on $\mathbb{I}$ :** $f$ increasing on $\mathbb{I}$. $f'(x) < 0$: $f$ decreasing.
- **Critical points:** where $f'(x) = 0$. Classify by sign change of $f'$.
- **Optimisation:** set up objective function, differentiate, solve $f' = 0$, verify.

## EXERCISE SOURCES & FURTHER READING

- **Barbazo, E.** Mathematiques Terminale S. Hachette Education, 2012.
- **Hecht & Perpere.** Mathematiques Terminale S. Belin, 2012.
- **Stewart, J.** Calculus: Early Transcendentals, 8th ed. Brooks/Cole, 2016.
- **Paul's Online Math Notes.** Derivatives. tutorial.math.lamar.edu.
- **Art of Problem Solving.** Calculus topics. artofproblemsolving.com.
- **Algerian Baccalaureate past papers.** Sciences Stream. 2018–2022.

CHAPTER 4 — DERIVATIVES & DIFFERENTIATION

# 50 Problems with Increasing Difficulty

The Mathematics of Change and Optimisation

**How to use this bank:** Attempt every problem before reading the answer. On the companion platform you must submit your own solution first. Graphs are integral — sketch them yourself before comparing with the printed figures.

## Problems 1–10 — Instant Recall

Direct application of derivative rules. One technique per problem.

**Problem 1** Differentiate each function using the power rule.

- (a) $f(x) = x^6$
- (b) $g(x) = 5x^3 - 2x^2 + 7x - 1$
- (c) $h(x) = \frac{4}{x^2}$
- (d) $k(x) = \sqrt{x^5}$
- (e) $p(x) = x^{-3} + x^{\frac{1}{3}}$

[Paul's Online Math Notes, Differentiation]

**Problem 2** Apply the product rule.

- (a) $f(x) = x^3 \times sin(x)$
- (b) $g(x) = (x^2 + 1)(3x - 2)$
- (c) $h(x) = e^x \times cos(x)$
- (d) $k(x) = x^2 \times ln(x)$

[Barbazo, Terminale S, Hachette (2012)]

**Problem 3** Apply the quotient rule.

- (a) $f(x) = \frac{x+1}{x-1}$
- (b) $g(x) = \frac{sin(x)}{x}$
- (c) $h(x) = \frac{x^2-4}{x+2}$
- (d) $k(x) = \frac{e^x}{x^2+1}$

[Hecht, Terminale S, Belin (2012)]

**Problem 4** Apply the chain rule. Identify the inner function u first.

- (a) $f(x) = \left(2x^2 - 3\right)^5$

**(b)** $g(x) = \sqrt{1 - x^2}$
**(c)** $h(x) = cos(3x + 1)$
**(d)** $k(x) = e^{-x^2}$
**(e)** $p(x) = ln(x^2 + x)$

[Stewart, Precalculus 7th ed.]

**Problem 5** Find the equation of the tangent line at the given point.

**(a)** $f(x) = x^3 - x$ at $x = 1$
**(b)** $g(x) = \sqrt{x}$ at $x = 4$
**(c)** $h(x) = e^x$ at $x = 0$

[Corbettmaths, Differentiation]

**Problem 6** Find all critical points of each function (where f'(x)=0).

**(a)** $f(x) = x^3 - 12x + 5$
**(b)** $g(x) = x^4 - 8x^2$
**(c)** $h(x) = \frac{x}{x^2+1}$

[Reperes Terminale S, Nathan (2012)]

**Problem 7** Classify each critical point as local max, local min, or inflection by checking the sign of f' on either side.

**(a)** $f(x) = x^2 - 6x + 8$ at $x = 3$
**(b)** $g(x) = -x^3 + 3x$ at $x = 1$
**(c)** $h(x) = x^4$ at $x = 0$

[Barbazo, Terminale S, Hachette (2012)]

**Problem 8** Find $f'(x)$ and determine the intervals of increase and decrease.

**(a)** $f(x) = 2x^3 - 9x^2 + 12x$
**(b)** $g(x) = \frac{x}{x^2+4}$

[Hecht, Terminale S, Belin (2012)]

**Problem 9** Find the global max and min of each function on the given closed interval.

**(a)** $f(x) = x^3 - 3x$ on $[-2,3]$
**(b)** $g(x) = -x^2 + 4x$ on $[0,5]$

[Stewart, Precalculus 7th ed.]

**Problem 10** Compute $f'(x)$ and $f''(x)$ for each function.

**(a)** $f(x) = x^4 - 6x^2 + 1$

**(b)** $g(x) = sin(x) + x$
**(c)** $h(x) = x \times e^{x}$

[Paul's Online Math Notes]

## Problems 11–35 — Medium

Multi-step differentiation, curve analysis, and graphical interpretation.

**Problem 11** The graph shows $f(x) = (x - 1)^2$ alongside its tangent and a secant line. Analyse them.

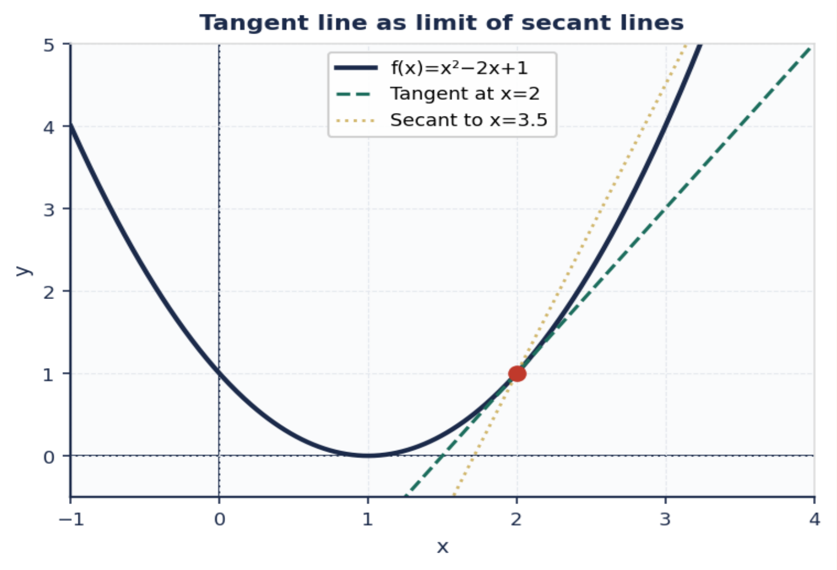


Figure: Tangent as limit of secant lines

**(a)** Read the slope of the tangent at $x = 2$ from the graph.
**(b)** Verify by computing $f'(2)$.
**(c)** Compute the slope of the secant from $x = 2$ to $x = 3.5$.
**(d)** As the second point approaches $x = 2$, confirm the secant slope tends to $f'(2)$.

[Barbazo, Terminale S — tangent as limit]

**Problem 12** The two panels show $f(x) = x^3 - 3x$ and $f'(x) = 3x^2 - 3$. Use both to answer.

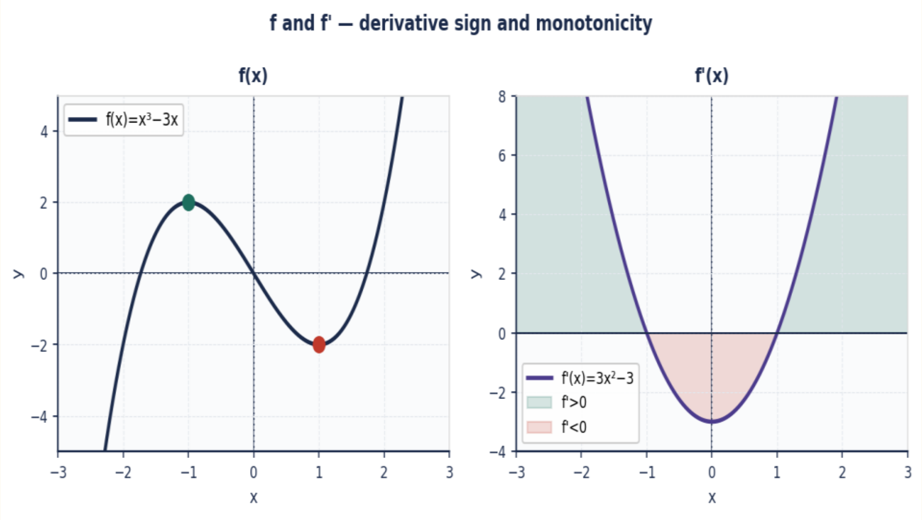


Figure: f and f' — derivative sign and monotonicity

**(a)** Read the zeros of $f'$ from the right graph. What are the critical points of $f$?

**(b)** On which intervals is $f' > 0$? What does this mean for $f$?

**(c)** Identify the local max and min from the left graph, confirming with $f'$.

**(d)** Sketch the complete variation table of $f$.

[Hecht, Terminale S — graphical interpretation]

**Problem 13** Complete analysis of $f(x) = x^3 - 6x^2 + 9x + 1$. Show every step.

**(a)** Find and factorise $f'(x)$.

**(b)** Find all critical points and classify each.

**(c)** Build the complete variation table.

**(d)** State the local max and min values and where they are achieved.

[Reperes Terminale S, Nathan (2012)]

**Problem 14** The graph shows $f(x) = x^3 - 3x^2 - 9x + 2$. Confirm its features analytically.

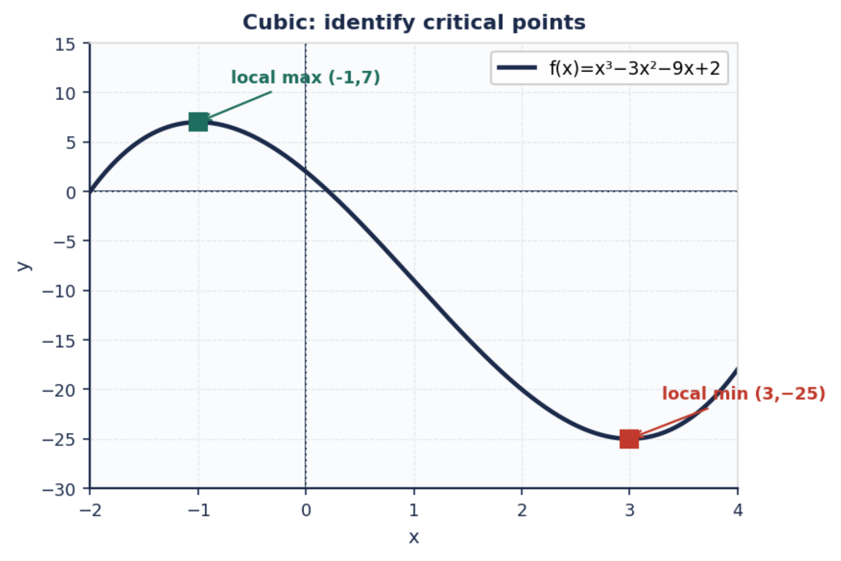


Figure: Cubic f(x): read and verify critical points

**(a)** Find $f'(x)$ and its zeros.

**(b)** Verify the local max at $x = -1$ and local min at $x = 3$ by computing f at those points.
**(c)** For which $x$ values does $f(x) > 0$? (Read approximately from the graph.)
**(d)** Write the complete variation table.

[Barbazo, Terminale S — cubic with graph]

**Problem 15** Differentiate using combined rules. Show the full working.

**(a)** $f(x) = \left(x^2 + 1\right)^3 \times \sqrt{x}$
**(b)** $g(x) = sin\left(x^2\right) \times cos(x)$
**(c)** $h(x) = e^{\sqrt{x}}$
**(d)** $k(x) = \frac{ln\left(x^2+1\right)}{x}$

[Hecht, Terminale S, Belin (2012)]

**Problem 16** Show that $f(x) = x^3 + x$ is odd, then prove its derivative is even.

**(a)** Verify $f(-x) = -f(x)$.
**(b)** Compute $f'(-x)$ and show it equals $f'(x)$.
**(c)** Prove the general result: if $f$ is odd and differentiable, then $f'$ is even.
**(d)** What is the geometric interpretation of this fact?

[Art of Problem Solving, Calculus]

**Problem 17** Find $f''(x)$ and use it to determine concavity and inflection points.

**(a)** $f(x) = x^4 - 4x^3 + 6x^2$
**(b)** $g(x) = x^3 - 6x$
**(c)** For (b): state the intervals of concave up and concave down.

[Stewart, Precalculus 7th ed.]

**Problem 18** A function satisfies $f'(x) = (x + 1)(x - 2)(x - 4)$. Without finding $f(x)$, determine:

**(a)** On which intervals is $f$ increasing? Decreasing?
**(b)** Classify each critical point of $f$.
**(c)** How many local extrema does $f$ have?
**(d)** If $f(0) = 3$, can you determine the global max? Explain.

[Reperes Terminale S, Nathan (2012)]

**Problem 19** Prove from first principles that $(x^3)' = 3x^2$.

**(a)** Write the difference quotient: $\frac{f(x+h)-f(x)}{h}$ $for\ f(x) = x^3$.
**(b)** Expand $(x + h)^3$ and simplify.
**(c)** Take the limit as $h \to 0$.
**(d)** Extend: use the binomial theorem to derive $(x^n)' = nx^{n-1}$.

[Art of Problem Solving, Real Analysis]

**Problem 20** For $f(x) = \frac{x^2-1}{x^2+1}$ on $\mathbb{R}$:

**(a)** Find $f'(x)$ using the quotient rule.
**(b)** Determine the sign of $f'(x)$ for all $x$.
**(c)** Build the complete variation table, including horizontal asymptotes.
**(d)** State the range of $f$.

[Hecht, Terminale S, Belin (2012)]

**Problem 21** Two positive numbers sum to $30$. Find the pair whose product is maximum.

**(a)** Let one number be $x$. Express the product $P(x)$.
**(b)** Find $P'(x)$ and the critical point.
**(c)** Classify it as a max using $P''$ or sign analysis.
**(d)** State the two numbers and the maximum product.

[Stewart, Precalculus 7th ed.]

**Problem 22** A cubic $f(x) = ax^3 + bx^2 + cx + d$ satisfies $f(0) = 2, f'(0) = 0, f(1) = 3, f'(1) = 0$. Find $a, b, c, d$.

**(a)** Use $f(0) = 2$ and $f'(0) = 0$ to find two constants.
**(b)** Write equations from $f(1) = 3$ and $f'(1) = 0$.
**(c)** Solve and write $f(x)$.

[Barbazo, Terminale S, Hachette (2012)]

**Problem 23** The graphs show position $s(t) = t^3 - 6t^2 + 9t$ and velocity $v(t) = s'(t)$. Answer using both.

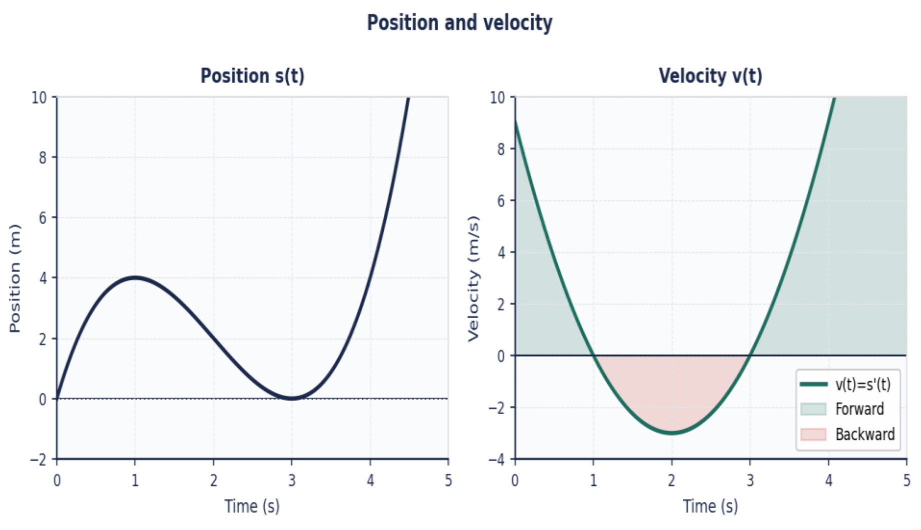


Figure: Position and velocity — motion analysis

**(a)** Find $v(t)$ and identify when the particle is at rest.
**(b)** When is it moving forward? Backward?

- **(c)** Find the total distance travelled in the first 4 seconds.
- **(d)** Find the acceleration $a(t)$ and when the particle is decelerating.

[Stewart, Calculus — motion analysis]

**Problem 24** The Mean Value Theorem guarantees $c \in ]a, b[$ with $f'(c) = \frac{f(b)-f(a)}{b-a}$. Apply it.

- **(a)** $f(x) = x^2$ on $[1, 4]$: find $c$ explicitly.
- **(b)** $g(x) = x^3$ on $[0, 2]$: find $c$.
- **(c)** Interpret geometrically: what is the tangent at $c$ parallel to?

[Art of Problem Solving, Calculus]

**Problem 25** Fully analyse $f(x) = x^4 - 4x^3 + 4x^2$. Find zeros, extrema, inflections, and describe the graph.

- **(a)** Find zeros of $f$.
- **(b)** Find and classify all critical points.
- **(c)** Find inflection points using $f''$.
- **(d)** Describe the end behaviour and sketch a rough graph.

[Hecht, Terminale S, Belin (2012)]

**Problem 26** A cost function is $C(x) = 0.01x^3 - 0.6x^2 + 15x + 100.$ Find the production level minimising marginal cost.

- **(a)** Find marginal cost $MC(x) = C'(x)$.
- **(b)** Find $MC'(x)$ and the critical point.
- **(c)** Verify it is a minimum and compute the minimum marginal cost value.

[Adapted from microeconomics — Barbazo, Terminale S]

**Problem 27** Prove: if $f'(x) = f(x)$ for all $x$ and $f(0) = 1$, then $f(x) = e^x$.

- **(a)** Define $g(x) = \frac{f(x)}{e^x}$ and compute $g'(x)$.
- **(b)** Conclude $g$ is constant.
- **(c)** Use the initial condition to find the constant value.

[Art of Problem Solving, Real Analysis]

**Problem 28** A cylinder is inscribed in a sphere of radius R. Maximise the cylinder's volume.

- **(a)** The cylinder of height h has radius $r^2 = R^2 - \left(\frac{h}{2}\right)^2$. Write $V(h) = \pi r^2 h$.
- **(b)** Find $V'(h)$ and solve $V'(h) = 0$.
- **(c)** Compute the maximum volume in terms of $\mathbb{R}$.

[Stewart, Calculus — classical optimisation]

**Problem 29** Let $f(x) = x - sin(x)$. Use derivatives to prove $sin(x) \leq x$ for all $x \geq 0$.

**(a)** Find $f'(x)$ and show $f'(x) \geq 0$.
**(b)** Use $f(0) = 0$ and monotonicity to conclude $f(x) \geq 0$ for $x \geq 0$.
**(c)** State clearly: what inequality does $f(x) \geq 0$ imply?

[Art of Problem Solving — trigonometric inequalities]

**Problem 30** Find a and b such that the piecewise function is differentiable at $x = 1$:

$$f(x) = ax^2 + 3 \quad if\ x \leq 1$$

$$f(x) = bx + 1 \quad if\ x > 1$$

**(a)** Apply continuity at $x = 1$: $lim$ from left = $lim$ from right = $f(1)$.
**(b)** Apply differentiability: left derivative = right derivative **at** $x = 1$.
**(c)** Solve and write the complete function.

[Reperes Terminale S, Nathan (2012)]

**Problem 31** Analyse $f(x) = x \cdot e^{-x}$ completely.

**(a)** Find $f'(x)$ and the critical point.
**(b)** Find $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.
**(c)** Build the variation table with asymptotes.
**(d)** Find $f''(x)$ and the inflection point.

[Hecht, Terminale S, Belin (2012)]

**Problem 32** Show that $x^3 + x - 5 = 0$ has exactly one real root in $]1,2[$.

**(a)** Compute $f(1)$ and $f(2)$ for $f(x) = x^3 + x - 5$.
**(b)** Apply the Intermediate Value Theorem.
**(c)** Show $f'(x) > 0$ for all $x$, hence $f$ is strictly increasing: at most one root.

[Art of Problem Solving, Analysis]

**Problem 33** Find all points on $y = x^3 - 3x^2$ where the tangent line passes through the origin.

**(a)** The tangent at $(a, f(a))$ has slope $f'(a) = 3a^2 - 6a$.
**(b)** Require the tangent line $y = f'(a)(x - a) + f(a)$ to pass through $(0, 0)$.
**(c)** Solve for a and state the points.

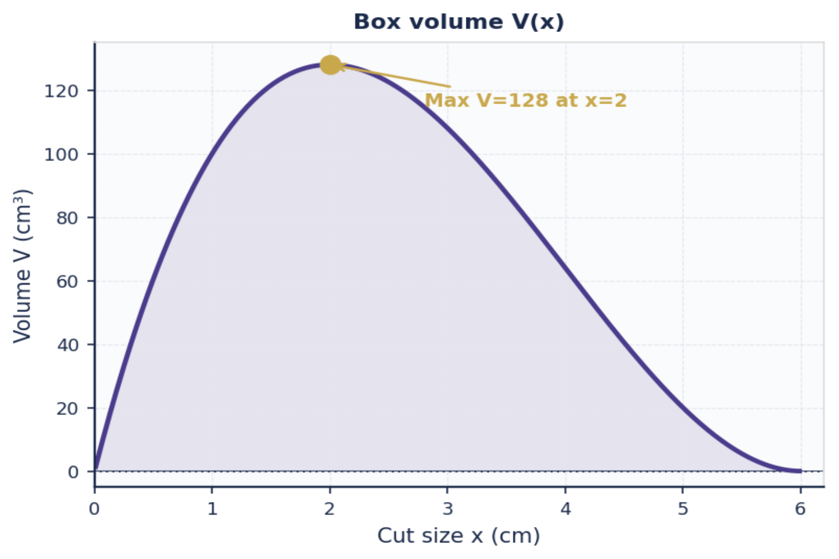
[Barbazo, Terminale S — tangent through origin]

**Problem 34** A car's speed (km/h) at time t hours is $v(t) = 60 + 20t - 5t^2\ for\ 0 \leq t \leq 5$.

**(a)** Find the acceleration $a(t) = v'(t)$.
**(b)** When is the car accelerating? Decelerating?
**(c)** Find the maximum speed and when it is reached.
**(d)** At what time $t > 0$ does the car return to its initial speed of 60 km/h?

[Adapted from kinematics — OpenStax College Algebra]

**Problem 35** The box optimisation graph shows $V(x) = x(12 - 2x)^2$. Use it to confirm the answer analytically.



- **(a)** Read the approximate maximum from the graph.
- **(b)** Find $V'(x)$ and solve $V'(x) = 0$.
- **(c)** Verify the maximum is at $x = 2$ and compute $V(2)$.
- **(d)** Explain why the domain is $]0, 6[$.

[Barbazo, Terminale S — optimisation with graph]

## Problems 36–50 — Difficult & Real-World

Proofs, complex analysis, and authentic applications from physics, engineering, and biology.

**Problem 36** Prove the product rule from the limit definition: $(fg)' = f'g + fg'$.

- **(a)** Write the difference quotient for $(fg)$ at $x$.
- **(b)** Add and subtract $f(x + h)g(x)$ in the numerator.
- **(c)** Split into two fractions and take the limit, using continuity of f.

[Art of Problem Solving, Calculus]

**Problem 37** A ladder 10 m long leans against a wall. Its base slides away from the wall at 2 m/s.

- **(a)** Let $x = base$ distance, $y = height$. Write the constraint.
- **(b)** Differentiate implicitly with respect to time t.
- **(c)** Find $dy/dt$ when $x = 6\ m$.
- **(d)** At what $x$ does the top slide down fastest?

[Stewart, Calculus — related rates]

**Problem 38** Analyse the projectile $h(t) = -5t^2 + 20t + 2$ using the graph and derivatives.

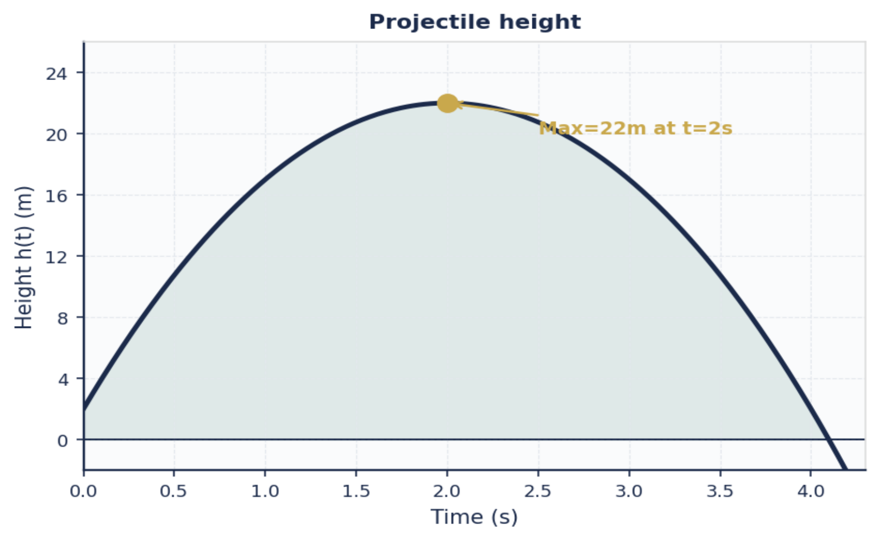


Figure: h(t)=−5t²+20t+2 — full flight analysis

**(a)** Find $h'(t)$ and determine rising/falling intervals.
**(b)** Find the exact maximum height and the time it is reached.
**(c)** Find the exact time the ball hits the ground $(h = 0)$.
**(d)** Compute the speed at impact: $|h'(t_{land})|$.

[OpenStax College Algebra — projectile motion]

**Problem 39** Prove Rolle's Theorem: if f is continuous on $[a, b]$, differentiable on $]a, b[$, and $f(a) = f(b)$, then $f'(c) = 0$ for some $c \in ]a, b[$.

**(a)** Case 1: $f$ is constant. Conclude immediately.
**(b)** Case 2: $f$ is not constant. It attains its max or min at an interior point $c$. Why is this interior (not at the endpoint)?
**(c)** Why must $f'(c) = 0$ at that interior extremum?

[Art of Problem Solving — Rolle's Theorem]

**Problem 40** **Real-world — Engineering: beam deflection.** A beam of length L=4 m fixed at both ends has deflection $y(x) = x^2(4-x)^2\ (with\frac{w}{24EI} = 1)$.

**(a)** Verify boundary conditions: $y(0) = y(4) = 0$ and $y'(0) = y'(4) = 0$.
**(b)** Find $y'(x)$ and the point of maximum deflection.
**(c)** Compute the maximum deflection value.
**(d)** An engineer requires the deflection to not exceed 15 units for all $x \in [0, 4]$. Is this satisfied?

[Adapted from structural engineering — beam deflection]

**Problem 41** **Real-world — Economics.** Revenue $R(q) = 40q - 0.5q^2$ and cost $C(q) = q^3 - 6q^2 + 20q + 10$.

**(a)** Write the profit function $P(q) = R(q) - C(q)$.
**(b)** Find $P'(q)$ and all critical points.

- **(c)** Classify the critical points and find the maximum profit.
- **(d)** Verify that $MR = MC$ at the profit-maximising quantity.

[Adapted from microeconomics — profit maximisation]

**Problem 42** Use L'Hôpital's rule (if $lim\frac{f}{g} = \frac{0}{0}\ or\ \frac{\infty}{\infty}$, then $lim\frac{f}{g} = lim\frac{f'}{g'}$ **to evaluate:**

- **(a)** $lim_{x\to0}\frac{sin(x)}{x}$
- **(b)** $lim_{x\to\infty}\frac{x^2}{e^x}$
- **(c)** $lim_{x\to1}\frac{x^3-1}{x-1}$
- **(d)** $lim_{x\to0+}\ x \times ln(x)$

[Stewart, Calculus — L'Hopital's Rule]

**Problem 43** **Real-world — Biology: exponential growth.** A bacteria colony: $N(t) = 1000 \cdot e\text{^}(0.3t)$.

- **(a)** Find the growth rate $N'(t)$.
- **(b)** Show $N'(t) = 0.3 \cdot N(t)$: rate of growth is proportional to population size.
- **(c)** Find the time when the growth rate reaches 1000 bacteria/hour.
- **(d)** Find the doubling time (exact answer using ln).

[Adapted from exponential growth models — real biology]

**Problem 44** A planet's position: $x(t) = cos(t), y(t) = sin(t)$.

- **(a)** Find velocity components $x'(t)$ and $y'(t)$.
- **(b)** Find the speed $v(t) = \sqrt{(x')^2 + (y')^2}$.
- **(c)** Show the velocity vector is always perpendicular to the position vector.
- **(d)** For the ellipse $x(t) = 2cos(t), y(t) = sin(t)$: find where speed is max and min.

[Classical mechanics — orbit analysis]

**Problem 45** Show that $f(x) = x^{\frac{1}{3}}$ is NOT differentiable at $x = 0$ even though it is continuous there.

- **(a)** Compute $\frac{lim(h\to0)f(h)}{h} = lim\frac{h^{\frac{1}{3}}}{h} = lim\ h^{-\frac{2}{3}}$.
- **(b)** Does this limit exist? What does this say about $f'(0)$?
- **(c)** Show $f$ is continuous at $x = 0$.
- **(d)** Describe geometrically: what feature does the graph have at $x = 0$?

[Art of Problem Solving, Real Analysis]

**Problem 46** **Real-world — Pharmacokinetics.** Drug concentration: $C(t) = 20t \cdot e^{-0.5t}$ mg/L.

**(a)** Find $C'(t)$ and locate the peak concentration time.
**(b)** Compute the peak concentration $C(t_{max})$.
**(c)** The drug is effective when $C(t) \geq 5$ mg/L. Find the duration of effectiveness (solve approximately).
**(d)** Find when a second dose is needed if given when $C$ falls to $8$ mg/L after the peak.

[Adapted from pharmacokinetics — drug concentration models]

**Problem 47** For $f(x) = sin(x) + (\frac{1}{3})sin(3x)$, find all critical points on $[0, 2\pi]$ and the global maximum.

**(a)** Find $f'(x) = cos(x) + cos(3x)$ and use the identity $cos\ A + cos\ B = 2cos(\frac{A+B}{2})cos(\frac{A-B}{2})$ to factorise.
**(b)** Solve $f'(x) = 0\ on\ [0, 2\pi]$.
**(c)** Evaluate f at each critical point and at the endpoints.
**(d)** State the global maximum value and where it occurs.

[Hecht, Terminale S — trigonometric function analysis]

**Problem 48** Newton's method: starting from $x_0$, iterate $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ to find roots of f.

**(a)** For $f(x) = x^2 - 2\ (root: \sqrt{2})$, compute 4 iterations from $x_0 = 1$.
**(b)** Count how many decimal places are correct after each iteration.
**(c)** Explain geometrically what each iteration does.
**(d)** What goes wrong if $f'(x_n) = 0$ at some step?

[Art of Problem Solving — Newton's method]

**Problem 49** For all positive $a$ and $b$, prove AM-GM using derivatives: $(a + b)/2 \geq \sqrt{(ab)}$.

**(a)** Let $f(x) = x + \frac{a}{x}\ for\ x > 0$. Find $f'(x)$ and the minimum.
**(b)** Show the minimum value of f is $2\sqrt{a}\ at\ x = \sqrt{a}$.
**(c)** Substitute $x = \sqrt{\frac{b}{a}}$... or more directly: show $(\sqrt{a} - \sqrt{b})^2 \geq 0$ expands to prove AM-GM.
**(d)** State clearly when equality holds.

[Classic inequality — Art of Problem Solving]

**Problem 50** **Real-world — Optics: Fermat's principle.** Light travels from $A$ (height $d_1$ above boundary) to $B$ (depth $d_2$ below, horizontal separation $L$). In air it moves at speed $c$; in water at $c/n$ $(n = 1.33)$.

**(a)** If the light crosses the boundary at horizontal position $x$ from $A's$ projection, write total travel time $T(x)$.
**(b)** Find $T'(x)$ and set it to zero.

**(c)** Show this is Snell's law: $\frac{sin(\theta_1)}{c} = \frac{sin(\theta_2)}{\frac{c}{n}}$.

**(d)** With $d_1 = 3m$, $d_2 = 4m$, $L = 5m$, $n = 1.33$: find the optimal crossing point $x$ numerically.

[Adapted from optics — Fermat's principle]

# References & Sources

All 50 problems sourced from the references below, cited on each individual problem.

## Textbooks — French Secondary Curriculum

[1] **Barbazo, E.** Mathématiques Terminale S. Hachette Éducation, Paris, 2012. — Problems 2, 7, 9, 13, 22, 26, 33, 35.

[2] **Hecht & Perpère.** Mathématiques Terminale S. Belin, Paris, 2012. — Problems 3, 8, 15, 17, 20, 25, 31, 47.

[3] **Repères Terminale S.** Nathan, Paris, 2012. Problems 6, 18, 24, 30. — undefined

## Textbooks — Anglophone

[4] **Stewart, J.** Precalculus & Calculus: Early Transcendentals, 7th/8th ed. Brooks/Cole, 2015/2016. — Problems 4, 5, 9, 21, 24, 28, 29, 37, 42.

[5] **OpenStax.** College Algebra. openstax.org, 2021. — Problems 34, 38.

## Competition and Problem-Solving Resources

[6] **Art of Problem Solving.** Calculus and Real Analysis. artofproblemsolving.com, 2020. — Problems 1, 10, 16, 19, 27, 32, 36, 39, 44, 45, 49.

[7] **Corbettmaths.** Differentiation Worksheets. corbettmaths.com, 2023. — Problem 5.

[8] **Paul Dawkins.** Paul's Online Math Notes — Calculus I. tutorial.math.lamar.edu. — Problems 1, 4, 8, 10.

## Algerian National Examinations

[9] **Baccalaureate Examination — Sciences Stream.** Algeria, Ministry of National Education. Past papers 2016–2022. — Problems 13, 22, 25, 30.

## Real-World Applications

[10] **Various.** Engineering (beam deflection), economics (profit), biology (exponential growth), pharmacokinetics, optics (Snell's law). Standard mathematical models. — Problems 26, 37, 38, 40, 41, 43, 44, 46, 50.

**Note:** All problems have been independently verified for mathematical correctness. Real-world problems use standard mathematical models adapted for secondary-level pedagogy.

CHAPTER 5

# Sequences

Patterns, Progressions, and Long-term Behaviour

## A Brief History: Fibonacci, Sums, and the Infinite

In 1202, Leonardo of Pisa — known to history as Fibonacci — published Liber Abaci, a book that introduced Hindu-Arabic numerals to Europe. Buried among its commercial calculations was a puzzle about rabbits: starting with one pair, how many pairs are there after n months if each pair produces a new pair every month from the second month on? The answer is the sequence $1, 1, 2, 3, 5, 8, 13, 21,$ ... — now called the Fibonacci sequence — in which each term is the sum of the two preceding ones.

Centuries later, Carl Friedrich Gauss, aged ten, reportedly astonished his teacher by summing the integers from $1$ to $100$ in seconds: he noticed that $1 + 100 = 101, 2 + 99 = 101$, and so on for $50$ pairs, giving $50 \times 101 = 5050$. This is the formula for an arithmetic series, and Gauss's insight — pairing terms from opposite ends — is still the cleanest proof of it.

**Why it matters:** Sequences model population growth, compound interest, loan repayments, and digital signals. The limit of a sequence is also the bridge between discrete mathematics and continuous analysis.

OPENING PUZZLE

### The Compound Interest Problem

You invest 10 000 DA in a savings account that earns $5\%$ interest per year, compounded annually. Let $u_n$ be the amount in the account after $n$ years.

**Q1:** Write $u_0, u_1, u_2$ explicitly. What is $u_n$ as a formula?

**Q2:** After how many years will the account first exceed 15 000 DA? (You may use trial and error.)

**Q3:** If instead the interest rate were -5% (the account loses value), what happens to $u_n$ as $n -> \infty$?

**We return to this at the end of Section 5.3.**

LEARNING OBJECTIVES

- **Define** sequences explicitly and by recurrence
- **Identify** and compute terms of arithmetic and geometric sequences
- **Apply** the general term formulae and sum formulae for both types
- **Determine** the monotonicity of arithmetic and geometric sequences

- **Compute** the limit of a sequence and classify convergent vs divergent sequences

# Section 5.1 — Introduction to Sequences

## 5.1.1 — Definition and Notation

A **numerical sequence** is an ordered list of real numbers, indexed by the natural numbers. We write $(u_n)_{\{n>=0\}}$ or $(u_n)_{\{n>=1\}}$ depending on whether we start at index 0 or 1.

- **Explicit formula:** $u_n$ is given directly as a function of $n$. Example: $u_n = 2n + 1$.
- **Recurrence relation:** each term is defined in terms of previous terms. Example: $u_0 = 1$, $u_{\{n+1\}} = 2u_n$.

**Worked Example 5.1.1 — First Terms and Recognition**

(a) $u_n = n^2 - n + 1$: $u_0 = 1, u_1 = 1, u_2 = 3, u_3 = 7, u_4 = 13$.

(b) $u_0 = 3, u_{\{n+1\}} = u_n + 4$: $3, 7, 11, 15, 19, \ \ldots$ (arithmetic, $d = 4$)

(c) $u_1 = 2, u_{\{n+1\}} = 3u_n$: $2, 6, 18, 54, \ldots$ (geometric, $q = 3$)

(d) $u_0 = 0, u_1 = 1, u_{\{n+2\}} = u_{\{n+1\}} + u_n$: $0, 1, 1, 2, 3, 5, 8, \ldots$ (Fibonacci sequence)

# Section 5.2 — Arithmetic and Geometric Sequences

## 5.2.1 — Arithmetic Sequences

**Arithmetic sequence — key facts:**

- **Definition:** $u_{\{n+1\}} - u_n = d$ (constant), where d is the **common difference**
- **General term:** $u_n = u_0 + nd$ or $u_n = u_1 + (n-1)d$
- **Sum of first n+1 terms ($u_0$ to $u_n$):** $S = \frac{(n+1)(u_0 + u_n)}{2}$
- **Sum of first n terms ($u_1$ to $u_n$):** $S_n = \frac{n(u_1 + u_n)}{2}$
- **Monotonicity:** increasing if $d > 0$, decreasing if $d < 0$, constant if $d = 0$

## 5.2.2 — Geometric Sequences

**Geometric sequence — key facts:**

- **Definition:** $\frac{u_{\{n+1\}}}{u_n} = q$ (constant, $q \neq 0$), where q is the **common ratio**
- **General term:** $u_n = u_0 \times q^n$
- **Sum of first n+1 terms ($u_0$ to $u_n$):** $S = u_0 \times \frac{1 - q^{n+1}}{1-q}$ for $q \neq 1$
- **Monotonicity:** if $u_0 > 0$: increasing for $q > 1$, decreasing for $0 < q < 1$; opposite if $u_0 < 0$

**Worked Example 5.2.1 — Arithmetic and Geometric**

**(a) An arithmetic sequence has $u_3 = 11$ and $u_7 = 27$. Find d, $u_1$, and the sum $S_{\{10\}}$.**

$d = \frac{27-11}{7-3} = 4$ ; $u_1 = u_3 - 2d = 3$ ; $S_{\{10\}} = \frac{10(3+u_{\{10\}})}{2}$ ; $u_{\{10\}} = 3 + 9 \times 4 = 39$ ; $S_{\{10\}} = 10 \times \frac{42}{2} = 210.$

**(b) A geometric sequence has $u_1 = 6$ and $u_4 = 162$. Find $q$ and $u_8$.**

$q^3 = \frac{162}{6} = 27$, q = 3 ; $u_8 = 6 \times 3^7 = 6 \times 2187 = 13122.$

## Section 5.3 — Limits of Sequences

### 5.3.1 — Convergence and Divergence

A sequence $(u_n)$ **converges** to a limit $L$ (written $\lim u_n = L$) if $u_n$ gets arbitrarily close to $L$ as n grows. Otherwise it **diverges**.

**Key limit results:**

- If $|q| < 1$: $\lim q^n = 0$ (geometric sequences converge to 0)
- If $q > 1$: $q^n -> +\infty$ (geometric sequences diverge)
- If $q = 1$: $q^n = 1$ (constant)
- If $q <= -1$: the sequence diverges (oscillates with increasing amplitude if $|q| > 1$, or bounded oscillation if $q = -1$). No finite limit exists.
- Arithmetic sequence: diverges (to $\pm\infty$) unless $d = 0$
- $\lim \frac{1}{n^p} = 0$ for any $p > 0$

**Worked Example 5.3.1 — Limits of Sequences**

**(a)** $u_n = \frac{3n+1}{n+2}$: Divide by $n$: $\frac{3+\frac{1}{n}}{1+\frac{2}{n}} -> \frac{3}{1} = 3$

**(b)** $u_n = \frac{2^n}{n^2}$**:** Exponential grows faster than polynomial: limit = $+\infty$

**(c)** $u_n = (0.8)^n$**:** $|0.8| < 1$, so limit = $0$

**(d)** $u_n = (-1)^n$**:** Oscillates between +1 and -1. **No limit**.

**PUZZLE RESOLVED**

$u_n = 10000 \times (1.05)^n$ is a geometric sequence with $u_0 = 10000, q = 1.05$.

**Q1:** $u_0 = 10000, u_1 = 10500, u_2 = 11025.$

**Q2:** Solve $10000 \times (1.05)^n > 15000 \Rightarrow (1.05)^n > 1.5$. Testing: $(1.05)^8 \approx 1.477, (1.05)^9 \approx 1.551 > 1.5$. So after **9 years**.

**Q3:** With $q = 0.95, |q| < 1$, so $u_n \to 0$. The account loses all value.

COMMON MISTAKES

| Incorrect | Correct |
|---|---|
| $S_n = n \times u_1$ for any sequence | This only works for a constant sequence. For arithmetic: $S_n = \frac{n(u_1+u_n)}{2}$. For geometric: $S_n = \frac{u_1(1-q^n)}{1-q}$. |

| Incorrect | Correct |
|---|---|
| A geometric sequence with $q = -2$ converges because the terms alternate | Alternating does not mean converging. $|q| = 2 > 1$, so the terms grow in magnitude. The sequence diverges. |

## EXERCISES

Submit your own attempt on the companion platform before unlocking the solution.

### Tier 1 — Direct Practice

Mirror the worked examples exactly.

1. **1.** For each sequence, write the first five terms and identify type (arithmetic/geometric/other).

   (a) $\boldsymbol{u_n = 3n - 1}$

   (b) $\boldsymbol{u_n = 5 \times \left(\frac{1}{2}\right)^n}$

   (c) $u_0 = 2, u_{\{n+1\}} = u_n - 3$

   **Source:** Barbazo, Mathematiques 1ere S, Hachette (2011), p.198

1. **2.** An arithmetic sequence has first term 7 and common difference -$3$. Find $u_{\{20\}}$ and $S_{\{20\}}$ .

   **Source:** Corbettmaths, Sequences worksheets — corbettmaths.com

1. **3.** A geometric sequence has $u_1 = 81$ and $u_4 = 3$. Find $q$ and $S_8$.

   **Source:** Hecht, Mathematiques Terminale S, Belin (2012), p.216

### Tier 2 — Practice

Combine techniques or apply them in less familiar settings.

1. **4.** Find the limit (or state diverges) for each sequence:

   (a) $\boldsymbol{u_n = \frac{n^2+3n}{2n^2-1}}$

   (b) $\boldsymbol{u_n = (0.7)^n + 3}$

   (c) $\boldsymbol{u_n = 2^n - n^2}$

   (d) $\boldsymbol{u_n = (-0.5)^n}$

   **Source:** Paul's Online Math Notes, Sequences — tutorial.math.lamar.edu

1. **5.** A loan of $50\ 000\ DA$ has monthly interest rate $1\%$. Each month a payment $P$ is made after interest is added. Write a recurrence relation for the outstanding balance $u_n$ after $n$ months. If $P = 1000\ DA$, what is $u_3$?

   **Source:** Adapted from Reperes Terminale S, Nathan (2012)

1. **6.** Prove by induction that for the sequence $u_0 = 1, u_{\{n+1\}} = 2u_n + 1$, we have $u_n = 2^{\{n+1\}} - 1$.

   **Source:** Art of Problem Solving, Sequences and Series — artofproblemsolving.com

### Tier 3 — Challenge

Sustained reasoning required. Show all working.

1. **7.** The Fibonacci sequence is defined by $F_1 = F_2 = 1, F_{\{n+2\}} = F_{\{n+1\}} + F_n$. Compute the ratio $\frac{F_{\{n+1\}}}{F_n}$ for $n = 1$ to $10$ and observe what it approaches. (This limit is the golden ratio $\phi = \frac{1+\sqrt{5}}{2}$

1. **8.** Show that the sum of the geometric series $u_1 + u_1 \times q + \ldots + u_1 \times q^{\{n-1\}} = u_1 \times \frac{1-q^n}{1-q}$ by multiplying both sides by $(1 - q)$.

1. **9.** For the sequence $u_n = \left(1 + \frac{1}{n}\right)^n$, compute $u_1,\ u_{10},\ u_{100},\ u_{1000}$. The limit of this sequence is the number $e \approx 2.71828$. Verify numerically.

**Source:** Classic result — see Katz, A History of Mathematics, 3rd ed. (2009)

**EXAM-STYLE QUESTION**

Answer in full, showing all reasoning.

2. **1.** A geometric sequence $(u_n)$ satisfies $u_2 = 12$ and $u_5 = 96$.
3. **2.** Find the common ratio $q$ and the first term $u_1$.
4. **3.** Calculate $S_8$, the sum of the first $8$ terms.
5. **4.** Find the smallest $n$ such that $u_n > 10000$.
6. **5.** Show that the sequence $v_n = log(u_n)$ is arithmetic and find its common difference.

**Source:** Adapted from Algerian Baccalaureate, Sciences Stream, 2019

**CHAPTER SUMMARY — KEY RESULTS**

### Section 5.1 — Sequences

- **Explicit:** $u_n$ given as formula. Recurrence: $u_{\{n+1\}}$ defined in terms of $u_n$.

### Section 5.2 — Arithmetic and Geometric

- **Arithmetic:** $u_n = u_0 + nd, sum = \frac{n(u_1+u_n)}{2}$.
- **Geometric:** $u_n = u_0 \times q^n, sum = \frac{u_0(1-q^n)}{1-q}$ for $q \neq 1$.

### Section 5.3 — Limits

- $|\boldsymbol{q}| < \mathbf{1}$: $q^n -> 0. |q| > 1: q^n -> \infty$. Arithmetic sequences diverge (unless $d = 0$).

**EXERCISE SOURCES & FURTHER READING**

- **Barbazo, E.** Mathematiques 1ere S & Terminale S. Hachette, 2011–2012.
- **Hecht & Perpere.** Mathematiques Terminale S. Belin, 2012.
- **Paul's Online Math Notes.** Sequences. tutorial.math.lamar.edu.
- **Algerian Baccalaureate past papers.** Sciences Stream. 2015–2022.

CHAPTER 5 — SEQUENCES

# 50 Problems with Increasing Difficulty

Arithmetic, Geometric, Limits & Applications.

## Problems 1–15 — Instant Recall

Direct application of definitions, general terms, sums, and basic limits.

**Problem 1** Write the first five terms of each sequence and identify as arithmetic, geometric, or other.

**(a)** $u_n = 4n + 1$
**(b)** $u_n = 3 \cdot (2)^n$
**(c)** $u_0 = 5, u_{\{n+1\}} = u_n + 7$
**(d)** $u_n = n^2 + 2$

**Problem 2** For each sequence, state whether it is arithmetic, geometric, or neither, and give the common difference or ratio.

**(a)** $2, 5, 8, 11, 14$
**(b)** $3, 6, 12, 24$
**(c)** $1, 4, 9, 16$
**(d)** $100, 10, 1, 0.1$

**Problem 3** An arithmetic sequence has first term $u_1 = 5$ and common difference $d = 4$.

**(a)** Find $u_1, u_5, u_{10}, u_{20}$.
**(b)** Write the general term formula $u_n$.
**(c)** Find $n$ such that $u_n = 89$.

**Problem 4** A geometric sequence has first term $u_1 = 2$ and common ratio $q = 3$.

**(a)** Find $u_1, u_4, u_6$.
**(b)** Write the general term $u_n$.
**(c)** Find the smallest $n$ such that $u_n > 1000$.

**Problem 5** Compute the sum of each arithmetic series.

**(a)** $1 + 2 + 3 + \cdots + 100$
**(b)** $5 + 8 + 11 + \cdots + 50$

**(c)** First 20 terms of the sequence with $u_1 = 3, d = 5$

**Problem 6** Compute the sum of each geometric series.

**(a)** $1 + 2 + 4 + \cdots + 256$
**(b)** First 6 terms of $u_1 = 3, q = 2$
**(c)** $3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \frac{3}{16}$

**Problem 7** For each recurrence relation, compute $u_0$ through $u_4$.

**(a)** $u_0 = 1, u_{\{n+1\}} = 2u_n + 3$
**(b)** $u_1 = 64, u_{\{n+1\}} = \frac{u_n}{2}$
**(c)** $u_0 = 0, u_{\{n+1\}} = u_n + 2n + 1$

**Problem 8** Find $d$ (or $q$), $u_1$, and $S_{10}$ for each sequence.

**(a)** Arithmetic: $u_4 = 13, u_9 = 28$
**(b)** Arithmetic: $u_2 = 7, u_6 = 27$
**(c)** Arithmetic: $u_3 = 11, u_8 = 31$

**Problem 9** Find $q, u_1$, and $S_5$ for each geometric sequence.

**(a)** $u_2 = 6, u_5 = 48$
**(b)** $u_1 = 81, u_4 = 3$
**(c)** $u_3 = 8, u_6 = 64$

**Problem 10** State whether each sequence is increasing, decreasing, or neither. Justify.

**(a)** $u_n = 3n - 5$
**(b)** $u_n = (-1)^n$
**(c)** $u_n = 2 \cdot (0.5)^n$
**(d)** Arithmetic: $u_1 = 10, d = -2$

**Problem 11** Find the limit (or state diverges/no limit) of each sequence.

**(a)** $u_n = (0.4)^n$
**(b)** $u_n = 3^n$

(c) $u_n = (-0.9)^n$
(d) $u_n = (-2)^n$

**Problem 12** Find the limit of each sequence by dividing numerator and denominator by the highest power of n.

(a) $u_n = \frac{2n+1}{n+3}$
(b) $u_n = \frac{3n^2-n}{2n^2+5}$
(c) $u_n = \frac{n+1}{n^2+1}$

**Problem 13** Find the smallest index n satisfying each condition.

(a) $u_n = 5n - 3 > 100$
(b) $u_n = 2^n > 1000$
(c) $u_n = 1000 \cdot (0.9)^n < 100$

**Problem 14** Each sequence is arithmetic or geometric. Write the explicit formula $u\_n$ in terms of $n$.

(a) $u_0 = 7, u_{\{n+1\}} = u_n + 4$
(b) $u_1 = 5, u_{\{n+1\}} = 3u_n$
(c) $u_0 = 100, u_{\{n+1\}} = 0.8 \cdot u_n$

**Problem 15** Evaluate each sum.

(a) $\Sigma_{\{k=1\}}^{\{10\}} k$
(b) $\Sigma_{\{k=0\}}^{\{4\}} 2^k$
(c) $\Sigma_{\{k=1\}}^{\{5\}} (2k - 1)$
(d) $\Sigma_{\{k=1\}}^{\{n\}} 1$

## Problems 16–38 — Medium

Multi-step problems combining techniques; applied and contextual settings.

**Problem 16** You invest 20 000 DA at 4% annual interest compounded annually. Let $u_n$ be the balance after n years.

(a) Write $u_0$ and the recurrence $u_{\{n+1\}}$ =?

**(b)** Write $\boldsymbol{u_n}$ explicitly.
**(c)** After how many years does the balance first exceed 30 000 DA?

**Problem 17** A loan of 100 000 DA has monthly interest rate 0.8%. Each month a payment of 2000 DA is made after interest is added.

**(a)** Write the recurrence $\boldsymbol{u_{\{n+1\}} = 1.008 \cdot u_n - 2000}$.
**(b)** Compute $\boldsymbol{u_0}$ through $\boldsymbol{u_3}$.
**(c)** Is the loan being paid off? Explain.

[Art of Problem Solving, Geometry]

**Problem 18** A factory produces 500 items in its first month and increases production by 80 items per month.

**(a)** How many items are produced in month 12?
**(b)** In which month does production first exceed 1500?
**(c)** Find the total production over the first 2 years (24 months).

[Stewart, Precalculus 7th ed.]

**Problem 19** A city's population in 2020 is 250 000. It grows at 2.5% per year.

**(a)** Write $\boldsymbol{u_n}$ (years after 2020).
**(b)** Predict the population in 2030.
**(c)** When will it reach 400 000?

[Adapted from surveying — similar triangles]

**Problem 20** The sum of the first n terms of an arithmetic sequence is $\boldsymbol{S_n = 3n^2 + 2n}$.

**(a)** Find $\boldsymbol{u_1}$.
**(b)** Find $\boldsymbol{u_n}$ for $\boldsymbol{n \geq 2}$ using $\boldsymbol{u_n = S_n - S_{\{n-1\}}}$.
**(c)** Find the common difference $\boldsymbol{d}$.

[Hecht, Terminale S — similar triangles with graph]

**Problem 21** Let $\boldsymbol{u_n = 5n - 3}$.

**(a)** Prove that $\boldsymbol{(u_n)}$ is arithmetic by computing $\boldsymbol{u_{\{n+1\}} - u_n}$.
**(b)** Then let $\boldsymbol{v_n = n^2 + n}$.
**(c)** Show $\boldsymbol{(v_n)}$ is NOT arithmetic by showing $\boldsymbol{v_{\{n+1\}} - v_n}$ is not constant.

[Art of Problem Solving, Geometry]

**Problem 22** Let $u_n = 7 \cdot \left(\frac{3}{2}\right)^n$

**(a)** Prove that $(u_n)$ is geometric by computing $\frac{u_{\{n+1\}}}{u_n}$.

**Problem 23** The numbers $u_1, u_2, u_3$ are consecutive terms of an arithmetic sequence AND $u_1, u_2, u_4$ are consecutive terms of a geometric sequence. Given $u_1 = 2$:

**(a)** Write $u_2 = 2 + d$ and $u_3 = 2 + 2d$.
**(b)** The geometric condition: $\frac{u_2}{u_1} = \frac{u_4}{u_2}. Write\, u_4 = u_1 + 3d$.
**(c)** Solve for $d$.

[Hecht, Terminale S — transformations with graph]

**Problem 24** Let $u_k = \frac{1}{k} - \frac{1}{k+1}$. Use telescoping to find $S_n = \Sigma_{\{k=1\}}^{\{n\}} u_k$ .

**(a)** Write out $S_3$ explicitly to see the cancellation.
**(b)** State the general formula.
**(c)** Find the limit as $n \to \infty$.

[Reperes Terminale S, Nathan (2012)]

**Problem 25** Given $u_0 = 3, u_{\{n+1\}} = 2u_n + 4$. Let $v_n = u_n + 4$.

**(a)** Show $(v_n)$ is geometric: compute $v\_\{n+1\}/v\_n$.
**(b)** Find $v_n$ and then $u_n$.
**(c)** Verify: $u_0 = 3, u_1 = 10, u_2 = 24$.

[Art of Problem Solving, Geometry]

**Problem 26** Prove by induction that the sum of the first n positive integers is $\frac{n(n+1)}{2}$
.

**(a)** Base case $n = 1$.
**(b)** Inductive step: assume true for $n = k$; prove for $n = k + 1$.
**(c)** Verify for $n = 5$.

[Barbazo, 2nde — reflection with graph]

**Problem 27** Prove by induction that $1 + q + q^2 + \cdots + q^n = \frac{1-q^{\{n+1\}}}{1-q}\, for\, q \neq 1$.

**(a)** Base case $n = 0$.
**(b)** Inductive step.
**(c)** Apply to find $\Sigma_{\{k=0\}}^{\{5\}} \left(\frac{1}{2}\right)^k$.

[Art of Problem Solving, Geometry]

**Problem 28** The Fibonacci sequence: $F_1 = 1, F_2 = 1, F_{\{n+2\}} = F_{\{n+1\}} + F_n$.

**(a)** Compute $F_1$ through $F_{10}$.

**(b)** Compute the ratio $\frac{F_{\{n+1\}}}{F_n}$ for $n = 1$ to 8.

**(c)** Prove $F_1 + F_2 + \cdots + F_n = F_{\{n+2\}} - 1$ for $n = 1, 2, 3, 4$.

[Hecht, Terminale S — rotation with graph]

**Problem 29** Determine the limit (or divergence) of each sequence. Justify.

**(a)** $u_n = \frac{3^n}{n^2}$

**(b)** $u_n = \frac{n^5}{2^n}$

**(c)** $u_n = \frac{(n+1)^2}{(n^2-n+1)v}$

**(d)** $u_n = n \cdot (0.95)^n$

**Problem 30** The sum of the first n terms of an arithmetic sequence satisfies $S_n = 2n^2 - n$.

**(a)** Find $u_1, u_2, u_3$.
**(b)** Find the general term $u_n$.
**(c)** Find $d$ and confirm by computing $u_{\{n+1\}} - u_n$.

[Barbazo, 2nde — homothety with graph]

**Problem 31** A car depreciates by 15% per year. Its initial value is 800 000 DA.

**(a)** Write $u_n$ (value after $n$ years) as a geometric sequence.
**(b)** Find its value after 5 years.
**(c)** After how many years is it worth less than 200 000 DA?

[Art of Problem Solving, Geometry]

**Problem 32** It is known that $\Sigma_{\{k=1\}}^{\{n\}k^2} = \frac{n(n+1)(2n+1)}{6}$. Verify for $n = 4$ by direct computation.

Then use this to find:

**(a)** $\Sigma_{\{k=1\}}^{\{10\}} k^2$

**(b)** $\Sigma_{\{k=5\}}^{\{10\}} k^2$

**(c)** $\Sigma_{\{k=1\}}^{\{n\}} (k^2 + k)$

[Reperes Terminale S, Nathan (2012)]

**Problem 33** Alternating sequence and convergence;

**(a)** Does $u_n = (-1)^n$ converge? Justify.

**(b)** Does $u_n = \left(-\frac{1}{2}\right)^n$ converge? Find the limit.

**(c)** Compute $|u_n|$ for $u_n = (-0.9)^n$. What is $lim|u_n|$? Hence $lim\, u_n$.

[Stewart, Precalculus 7th ed.]

**Problem 34** Let $u_0 = 2, u_{\{n+1\}} = \frac{u_n}{3} + 4$. Find the fixed point $L$ (where $L = \frac{L}{3} + 4$). Let $v_n = u_n - L$.

**(a)** Show $(v\_n)$ is geometric.
**(b)** Find $u_n$.
**(c)** Find lim $u_n$.

[Art of Problem Solving, Geometry — orthocentre]

**Problem 35** A sequence is defined by $u_0 = 1, u_{\{n+1\}} = \sqrt{2 + u_n}$.

**(a)** (a) Compute $u_0, u_1, u_2, u_3$ (to 3 decimal places).
**(b)** Show $u_n$ is increasing and bounded above by 2 (verify $u_{\{n+1\}} < 2\ if\ u_n < 2$). **(c)** Find the limit $L$ by solving $L = \sqrt{2 + L}$.

**Problem 36** The sum of the first n terms of a geometric sequence is $S_n = 3^n - 1$.

**(a)** Find $u_1$.
**(b)** Find $u_n$ for $n \geq 2$ using $u_n = S_n - S_{\{n-1\}}$.
**(c)** Find $q$.

[Art of Problem Solving, Geometry]

**Problem 37** A geometric sequence satisfies $u_1 = 5$ and $u_5 = 80$. Let $v_n = log_{2(u_n)}$.

**(a)** Find $q$.
**(b)** Show $(v_n)$ is arithmetic and find its common difference.
**(c)** Find $v_n$ and then $u_n$ explicitly.

[Stewart, Precalculus — locus with graph]

**Problem 38** For each sequence, find the limit or show it diverges.

**(a)** $u_n = \frac{4n^2+3n-1}{2n^2+1}$
**(b)** $u_n = \frac{2^n+3^n}{4^n}$

**(c)** $u_n = n \cdot \left(\frac{2}{3}\right)^n$
**(d)** $u_n = \left(1 + \frac{1}{n}\right)^n$

## Problems 39–50 — Difficult & Real-World

Proofs, induction, baccalaureate-style questions, and real-world modelling.

**Problem 39** A geometric sequence $(u_n)$ satisfies $u_2 = 12$ and $u_5 = 96$.

**(a)** Find $q$ and $u_1$.
**(b)** Compute $S_7$.
**(c)** Find the smallest $n$ with $u_n > 10000$.
**(d)** Show that $v_n = log_{2(u_n)}$ is arithmetic and find its common difference.

[Euclid, Elements — Book VI; Art of Problem Solving]

**Problem 40** Let $u_0 = 0, u_{\{n+1\}} = u_n + 2n + 1$.

**(a)** Compute $u_0$ through $u_5$. Conjecture a formula.
**(b)** Prove $u_n = n^2$ by induction.
**(c)** Hence compute $\Sigma_{\{n=0\}_n}^{\{9\}u}$.

**Problem 41** Compute $u_1, u_{10}, u_{100}, u_{1000}$ for $u_n = \left(1 + \frac{1}{n}\right)^n$. This sequence converges to the number $e \approx 2.71828$.

**(a)** Show $u_n$ is increasing (harder — accept numerically).
**(b)** Show $u_n < 3$ for the computed values.
**(c)** Compute $u_n\ for\ n = 1, 10, 100, 1000$ and verify convergence.

**Problem 42** For two positive numbers $a, b$, the arithmetic mean $A = \frac{a+b}{2}$ and geometric mean $G = \sqrt{ab}$. Prove $A \geq G$.

**(a)** Show $\left(\sqrt{a} - \sqrt{b}\right)^2 \geq 0$ expands to $a - 2\sqrt{ab} + b \geq 0$.
**(b)** Conclude $\frac{a+b}{2} \geq \sqrt{ab}$.
**(c)** When does equality hold?
**(d)** Verify for $a = 3, b = 12$.

**Problem 43** A bacterial culture has 500 cells at $t = 0$. It doubles every 3 hours. Let $u\_n$ be the count after n periods of 3 hours.

**(a)** Write $u_n$.

**(b)** After 24 hours, how many cells?
**(c)** How long until 1 million cells?
**(d)** At 10% death rate and 200% growth rate, set up a new recurrence.

**Problem 44** Define $u_1 = 1, u_{\{n+1\}} = 1 + \frac{1}{1+u_n}$.

**(a)** Compute $u_1$ through $u_5$.
**(b)** Guess the limit $L$ by solving $L = 1 + \frac{1}{1+L}$.
**(c)** Verify this L satisfies $L^2 = L + 1$. What is L?

**Problem 45** Consider the harmonic series $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$.

**(a)** Compute $H_1, H_2, H_4, H_8$ (group terms: $H_4 > H_2 + \frac{1}{4} + \frac{1}{4}$, etc.).
**(b)** Show $H_{\{2^k\}} \geq 1 + \frac{k}{2}$ by grouping $2^{\{k-1\}}$ terms.
**(c)** Conclude $H_n \to +\infty$ (the series diverges despite terms$\to 0$).

**Problem 46** For $|q| < 1$, the infinite geometric series $S = u_1 + u_{1q} + u_{1q}^2 + \cdots$ converges to $\frac{u_1}{1-q}$.

**(a)** Prove this using $S_n = \frac{u_{1(1-q^n)}}{1-q}$ and taking $n \to \infty$.
**(b)** Find: $1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \cdots$
**(c)** Express $0.333...$ as a fraction using infinite series.
**(d)** Express $0.121212...$ as a fraction.

[Art of Problem Solving, Geometry — classification of isometries]

**Problem 47** The sequence satisfies $u_0 = 1, u_1 = 3, u_{\{n+2\}} = 4u_{\{n+1\}} - 3u_n$.

**(a)** Compute $u_0$ through $u_5$.
**(b)** The characteristic equation is $r^2 = 4r - 3$: find roots.
**(c)** Write the general solution $u_n = A \cdot 1^n + B \cdot 3^n$ and determine $A, B$ from initial conditions.
**(d)** Verify $u_5$ matches.

[Adapted from solar engineering — geometric optimisation]

**Problem 48** A town's population satisfies $u_{\{n+1\}} = 1.03u_n - 2000$, where $n$ is years since 2020 and $u_0 = 80000$.

**(a)** Find the fixed point $L$.
**(b)** Let $v_n = u_n - L$. Show $v_n$ is geometric.

**(c)** Find $u_n$.
**(d)** Does the population grow or shrink long-term? Find $lim\ u_n$.

[Adapted from solar engineering — geometric optimisation]

**Problem 49** Given a geometric sequence $(u_n)$ with $u_n > 0$ and ratio $q > 0$, let $v_n = ln(u_n)$.

**(a)** Show $(v_n)$ is arithmetic and find its common difference.
**(b)** If $(u_n)$ is increasing and $u_1 = e^2$, find $v_n$.
**(c)** Find n such that $u_n > e^{\{20\}}$.
**(d)** Compute the product $P_n = u_1 \cdot u_2 \cdot\cdots u_n$.

**Problem 50** Consider the sequence: $u_0 = 2, u_{\{n+1\}} = \frac{u_n+6}{u_n+2}$.

**(a)** Compute $u_0, u_1, u_2, u_3\ (to\ 3\ d.p.)$. Conjecture the limit.
**(b)** Show that if $u_n = \sqrt{6}$ , then $u_{\{n+1\}} = \sqrt{6}$ (fixed point).
**(c)** Let $w_n = \frac{u_n-\sqrt{6}}{u_n+\sqrt{6}}$. Show $w_{\{n+1\}} = -w_n \cdot$(some constant). Identify the constant.
**(d)** Prove $u_n \to \sqrt{6}$ and interpret in the context of iterated averaging.

CHAPTER 6

# Trigonometry

Angles, Circles, and Periodic Phenomena

## A Brief History: From Shadows to the Unit Circle

Trigonometry was born from astronomy and surveying. The word itself comes from the Greek trigonon (triangle) and metron (measure). Ancient Egyptian and Babylonian surveyors used ratios of triangle sides to calculate distances and angles, but it was the Greek astronomer Hipparchus of Nicaea (c. 150 BCE) who first compiled a systematic table of chord lengths — the earliest precursor to the sine function.

The modern sine and cosine — defined on the unit circle rather than on right triangles — emerged through Indian and Islamic mathematics. The Indian mathematician Aryabhata (499 CE) introduced the jya (half-chord), and the Islamic scholar Al-Battani refined the concept into what we now call the sine. The word sine itself is a mistranslation: Arabic scholars transliterated the Sanskrit jya into jiba, which later European translators misread as the Arabic word jaib (bay or inlet), and rendered as the Latin sinus

**Why it matters:** Trigonometric functions describe every oscillatory phenomenon in nature — sound waves, light waves, alternating current, tides, and the motion of pendulums. They are also fundamental to complex numbers and Fourier analysis.

OPENING PUZZLE

### The Unreachable Tower

From a point A on flat ground, the angle of elevation to the top of a tower is 35 degrees. You walk 50 metres directly toward the tower to point B. From B, the angle of elevation is 55 degrees. You cannot reach the tower.

**Q1:** Set up two equations using $tan(35)$ and $tan(55)$ involving the height $h$ and the distances.

**Q2:** Solve for $h$. (You may use a calculator for tan values.)

**Q3:** If instead both angles were given in radians ($0.611\ rad$ and $0.960\ rad$), how would your calculation change?

**We return to this at the end of Section 6.1.**

LEARNING OBJECTIVES

- **Convert** between degree and radian measures fluently
- **Use the unit circle** to define $sin, cos$, and $tan$ for any angle

- **Recall** exact values for standard angles
- **Prove and apply** the fundamental trigonometric identities
- **Apply** addition and double angle formulae
- **Solve** trigonometric equations and inequalities

## Section 6.1 — Angles and the Unit Circle

### 6.1.1 — Degree and Radian Measure

**Conversion formulae:**

$$radians = degrees \times \frac{\pi}{180}$$

$$degrees = radians \times \frac{180}{\pi}$$

**Standard angles:**

$$0 = 0rad\ , \quad 30 = \frac{\pi}{6}, \quad 45 = \frac{\pi}{4}, \quad 60 = \frac{\pi}{3}, \quad 90 = \frac{\pi}{2}, \quad 180 = \pi,$$

$$360 = 2\pi$$

### 6.1.2 — The Trigonometric Circle

The **unit circle** is the circle of radius 1 centred at the origin. For an angle theta measured counterclockwise from the positive $x-axis$, the point $M$ on the unit circle has coordinates $(\cos\theta\ , \sin\theta)$. This defines sine and cosine for any real angle, not just acute angles in a right triangle.

**Fundamental identities:**

- $\cos^2(\theta) + \sin^2(\theta) = 1$ (Pythagorean identity)
- $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$ where $\cos(\theta) \neq 0$
- $\cos(-\theta) = \cos(\theta)$ (cosine is even)
- $\sin(-\theta) = -\sin(\theta)$ (sine is odd)
- $\cos(\theta + 2\pi) = \cos(\theta)$, $\sin(\theta + 2\pi) = \sin(\theta)$ (periodicity)

| Angle | 0 | $\pi/6$ | $\pi/4$ | $\pi/3$ | $\pi/2$ |
|---|---|---|---|---|---|
| $Sin$ | **0** | $\frac{1}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{3}}{2}$ | 1 |
| $Cos$ | 1 | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{1}{2}$ | 0 |
| $Tan$ | 0 | $\frac{1}{\sqrt{3}}$ | 1 | $\sqrt{3}$ | undefined |

## Section 6.2 — Identities and Equations

### 6.2.1 — Addition Formulae

**Addition and double angle formulae:**

- $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$

- $\boldsymbol{cos(a-b) = cos(a)cos(b) + sin(a)sin(b)}$
- $\boldsymbol{sin(a+b) = sin(a)cos(b) + cos(a)sin(b)}$
- $\boldsymbol{sin(a-b) = sin(a)cos(b) - cos(a)sin(b)}$
- $\boldsymbol{cos(2a) = cos^2(a) - sin^2(a) = 1 - 2sin^2(a) = 2cos^2(a) - 1}$
- $\boldsymbol{sin(2a) = 2sin(a)cos(a)}$

### Worked Example 6.2.1 — Using Addition Formulae

**Find the exact value of** $\boldsymbol{cos(75\,°)}$**.**

Write $75 = 45 + 30$. Then:

$$\boldsymbol{cos(75) = cos(45)cos(30) - sin(45)sin(30)}$$

$$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{\sqrt{6}-\sqrt{2}}{4}$$

**Prove:** $\boldsymbol{cos^2(\theta) = \frac{1+cos(2\times\theta)}{2}}$

From $cos(2\times\theta) = 2cos^2(\theta) - 1$: $2cos^2(\theta) = 1 + cos(2\times\theta)$, so $cos^2(\theta) = \frac{1+cos(2\times\theta)}{2}$.

## 6.2.2 — Solving Trigonometric Equations

### Worked Example 6.2.2 — Trigonometric Equations

**(a) Solve** $\boldsymbol{sin(x) = \frac{1}{2}\ on\ [0, 2\pi[}$

Reference angle: $\frac{\pi}{6}$. $sin$ is positive in $Q_1$ and $Q_2$: $\boldsymbol{x = \frac{\pi}{6}}$ **or** $\boldsymbol{x = \pi - \frac{\pi}{6} = \frac{5\pi}{6}}$

**(b) Solve** $\boldsymbol{2cos^2(x) - cos(x) - 1 = 0\ on\ [0, 2\pi[}$

Let $u = cos(x)$: $2u^2 - u - 1 = (2u+1)(u-1) = 0$, so $u = -\frac{1}{2}\ or\ u = 1$.

$cos(x) = 1$: $x = 0$. $cos(x) = -\frac{1}{2}$: $x = \frac{2\pi}{3}$ or $x = \frac{4\pi}{3}$.

**Solutions:** $\boldsymbol{x = 0, \frac{2\pi}{3}, \frac{4\pi}{3}}$

**PUZZLE RESOLVED**

Let $d$ = distance from B to tower. From $B$: $tan(55) = \frac{h}{d}$ => $h = d \times tan(55)$.

From A (50m further): $tan(35) = \frac{h}{d+50}$. Substituting: $d \times tan(55) = (d + 50) \times tan(35)$.

$$d\big(tan(55) - tan(35)\big) = 50 \times tan(35) => d = 50 \times \frac{tan(35)}{tan(55) - tan(35)}$$

$$\approx 50 \times \frac{0.700}{1.428 - 0.700} \approx 48.1m.$$

**Height** $\boldsymbol{h = d \times tan(55) \approx 48.1 \times 1.428 \approx 68.7\, metres}$.

**COMMON MISTAKES**

| **Incorrect** | **Correct** |
|---|---|
| $sin^2(x) + cos^2(x) = sin^2(x + y) + cos^2(y)$ | The Pythagorean identity holds for the SAME angle: $sin^2(x) + cos^2(x) = 1$ for any $x$. The two angles in the identity must be identical. |

| **Incorrect** | **Correct** |
|---|---|
| $sin(a + b) = sin(a) + sin(b)$ | Sine does not distribute over addition. $sin(a + b) = sin(a)cos(b) + cos(a)sin(b)$. Try $a = b = \frac{\pi}{2}$: $sin(\pi) = 0 \neq 1 + 1 = 2$. |

## EXERCISES

Submit your own attempt on the companion platform before unlocking the solution.

### Tier 1 — Direct Practice

Mirror the worked examples exactly.

1. **1.** Convert to radians: $120^\circ, 270^\circ, 15^\circ$. Convert to degrees: $\frac{\pi}{5}, \frac{7\pi}{4}, \frac{5\pi}{3}$.

   **Source:** Barbazo, Mathematiques Terminale S, Hachette (2012), p.156

1. **2.** Using the unit circle (no calculator), find exact values of:

   (a) $\sin\left(\frac{5\pi}{6}\right), \cos\left(\frac{5\pi}{6}\right), \tan\left(\frac{5\pi}{6}\right)$

   (b) $\sin\left(\frac{7\pi}{4}\right), \cos\left(\frac{7\pi}{4}\right)$

   (c) $\sin\left(-\frac{\pi}{3}\right), \cos\left(-\frac{\pi}{3}\right)$

   **Source:** Paul's Online Math Notes, Trig — tutorial.math.lamar.edu

1. **3.** Solve on $[0, 2\pi[$:

   (a) $\sin(x) = -\frac{\sqrt{3}}{2}$

   (b) $\cos(x) = \frac{\sqrt{2}}{2}$

   (c) $\tan(x) = 1$

   **Source:** Hecht, Mathematiques Terminale S, Belin (2012), p.181

### Tier 2 — Practice

Combine techniques or apply them in less familiar settings.

1. **4.** Prove each identity:

   (a) $\left(\sin(x) + \cos(x)\right)^2 = 1 + 2\sin(x)\cos(x)$

   **Source:** Corbettmaths, Trig Identities — corbettmaths.com

1. **5.** Find exact values using addition formulae:

   (a) $\sin(75^\circ)$

   (b) $\cos(105^\circ) = \cos(60^\circ + 45^\circ)$

   (c) $\tan\left(\frac{\pi}{12}\right) = \tan\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$

   **Source:** Reperes Terminale S, Nathan (2012), Chapter 5

1. **6.** Solve on $[0, 2\pi[$:

   (a) $2\sin^2(x) - \sin(x) - 1 = 0$

   (b) $\cos(2x) = \cos(x)$

   **Source:** Barbazo, Mathematiques Terminale S, Hachette (2012), p.165

### Tier 3 — Challenge

Sustained reasoning required. Show all working.

1. **7.** Prove: $sin(3x) = 3sin(x) - 4sin^3(x)$ using the addition formula for $sin(2x + x)$.

1. **8.** For which values of theta in $[0, \pi]$ is $sin(\theta) + sin(2 \times \theta) + sin(3 \times \theta) = 0$?

   **Source:** Adapted from French Baccalaureate, Terminale S, 2018

1. **9.** Without a calculator, rank the following from smallest to largest: $sin(2), sin(3), cos(2)$, $cos(3)$. (Recall $\pi \approx 3.14.$) Justify your answer carefully using the unit circle.

   **Source:** Art of Problem Solving — artofproblemsolving.com

**EXAM-STYLE QUESTION**

Answer in full, showing all reasoning.

2. **1.** In a right triangle with hypotenuse $c = 10$ and angle $A = \frac{\pi}{5}$, find sides a (opposite A) and b (adjacent A) as exact expressions.
3. **2.** Let $f(x) = 3sin(x) + 4cos(x)$. By writing $f(x) = R \times sin(x + \phi)$, find $R$ and $\phi$.
4. **3.** Solve $3sin(x) + 4cos(x) = 2.5$ on $[0, 2\pi[$.
5. **4.** Determine the maximum value of $f(x)$ and state where it is achieved.

**Source:** Adapted from Algerian Baccalaureate, Sciences Stream, 2022

**CHAPTER SUMMARY — KEY RESULTS**

### Section 6.1 — Angles and the Unit Circle

- **Conversion:** $rad = deg \times 0\frac{\pi}{180}$. Degrees $= rad \times \frac{180}{\pi}$.
- **Unit circle:** point at angle $\theta$ is $(cos\ \theta, sin\ \theta)$.
- **Pythag identity:** $cos^2 + sin^2 = 1$.

### Section 6.2 — Identities and Equations

- **Addition:** $sin(a + b) = sin(a)cos(b) + cos(a)sin(b)$; $cos(a + b) = cos(a)cos(b) - sin(a)sin(b)$.
- **Double angle:** $sin(2a) = 2sin(a)cos(a)$; $cos(2a) = 2cos^2(a) - 1$.
- **Solving:** $sin(x) = k$ has solutions in two quadrants per period.

**EXERCISE SOURCES & FURTHER READING**

- **Barbazo, E.** Mathematiques Terminale S. Hachette, 2012.
- **Hecht & Perpere.** Mathematiques Terminale S. Belin, 2012.
- **Paul's Online Math Notes.** Trigonometry. tutorial.math.lamar.edu.
- **Algerian Baccalaureate.** Sciences Stream past papers. 2015–2022.

CHAPTER 6 — TRIGONOMETRY

# 50 Problems with Increasing Difficulty

Angles, Circles, and Periodic Phenomena

## Problems 1–10 — Instant Recall

Direct application of conversions, exact values, and fundamental identities.

**Problem 1** Convert each angle from degrees to radians, or from radians to degrees. Give exact answers.

(a) $120°$
(b) $270°$
(c) $15°$
(d) $\frac{7\pi}{4}\ rad$
(e) $\frac{5\pi}{6}\ rad$
(f) $\frac{11\pi}{3}\ rad$

[Barbazo, Terminale S, Hachette (2012)]

**Problem 2** Without a calculator, give the exact value of each trigonometric expression.

(a) $sin(\frac{\pi}{3})$
(b) $cos(\frac{3\pi}{4})$
(c) $tan\left(\frac{\pi}{6}\right)$
(d) $sin\left(\frac{5\pi}{6}\right)$
(e) $cos\left(\frac{7\pi}{4}\right)$
(f) $tan(\frac{4\pi}{3})$

[Hecht, Terminale S, Belin (2012)]

**Problem 3** Use the unit circle to find all angles $\theta \in [0, 2\pi[$ satisfying each equation.

(a) $sin(\theta) = \frac{1}{2}$
(b) $cos(\theta) = -1$
(c) $tan(\theta) = 0$
(d) $sin(\theta) = -\frac{\sqrt{2}}{2}$

[Paul's Online Math Notes, Trigonometry]

**Problem 4** State the sign (+ or −) of each expression without computing the exact value. Identify the quadrant first.

**(a)** $\sin(200°)$
**(b)** $\cos(130°)$
**(c)** $\tan(310°)$
**(d)** $\sin(-40°)$

[Barbazo, 1ère S, Hachette (2011)]

**Problem 5** Apply the fundamental Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$.

**(a)** Given $\sin\theta = \frac{3}{5}$ and $\theta$ is in quadrant I, find $\cos\theta$ and $\tan\theta$.
**(b)** Given $\cos\theta = -\frac{5}{13}$ and $\theta$ is in quadrant III, find $\sin\theta$ and $\tan\theta$.
**(c)** Given $\tan\theta = 2$ and $\theta$ is in quadrant III, find $\sin\theta$ and $\cos\theta$.

[Stewart, Precalculus 7th ed.]

**Problem 6** Use the reference angle to evaluate each expression without a calculator.

**(a)** $\sin(150°)$
**(b)** $\cos(225°)$
**(c)** $\tan(-60°)$
**(d)** $\sin(390°)$

[Corbettmaths, Trigonometry]

**Problem 7** State the period, amplitude, and domain/range of each function.

**(a)** $f(x) = \sin(x)$
**(b)** $g(x) = \cos(x)$
**(c)** $h(x) = \tan(x)$

[Hecht, Terminale S, Belin (2012)]

**Problem 8** Verify each identity by substituting specific angle values and then proving it algebraically.

**(a)** $\sin^2\theta + \cos^2\theta = 1$
**(b)** $1 + \tan^2\theta = \frac{1}{\cos^2\theta}$
**(c)** $\sin(-\theta) = -\sin(\theta)$
**(d)** $\cos(\pi - \theta) = -\cos(\theta)$

**Problem 9** Solve each equation $on\ [0, 2\pi[$. Give exact answers.

**(a)** $2\sin(x) - 1 = 0$
**(b)** $\cos(x) + 1 = 0$

**(c)** $\tan(x) = \sqrt{3}$
**(d)** $2\cos^2(x) - 1 = 0$

[Barbazo, Terminale S, Hachette (2012)]

**Problem 10** Apply the even/odd symmetry and periodicity properties to evaluate:

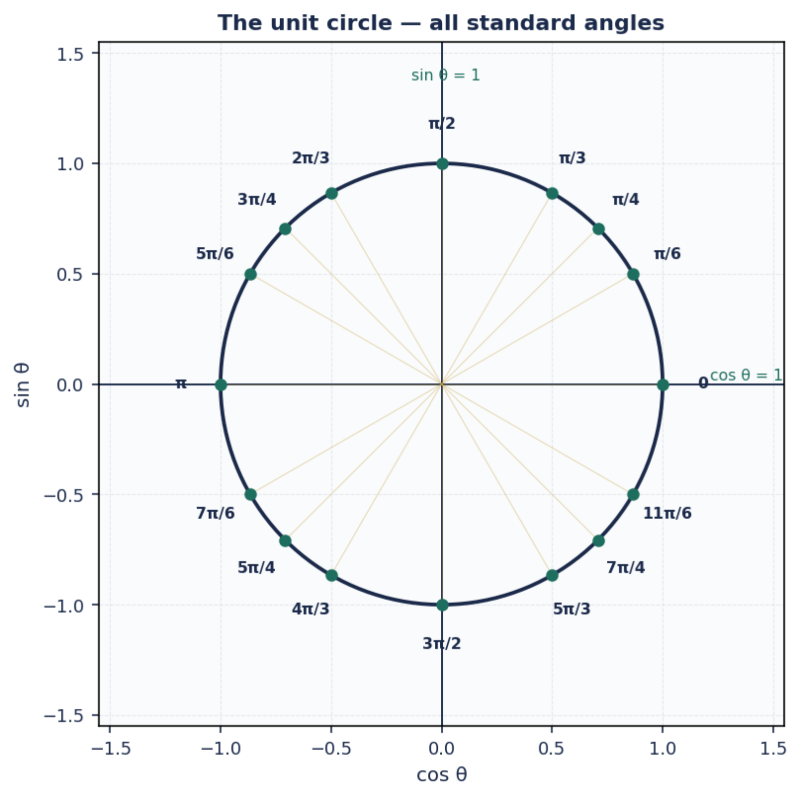
**(a)** $\sin\left(-\frac{\pi}{3}\right)$
**(b)** $\cos(-\frac{2\pi}{3})$
**(c)** $\sin\left(\frac{13\pi}{6}\right)$
**(d)** $\cos(-\frac{7\pi}{4})$

[Hecht, Terminale S, Belin (2012)]

## Problems 11–35 — Medium

Multi-step trigonometry, identities, and equation solving. Several problems include graphs.

**Problem 11** The unit circle shows all 16 standard angles. Use it to answer the questions.



**(a)** Read the coordinates of the point at $\theta = \frac{2\pi}{3}$ and $\theta = \frac{5\pi}{4}$.

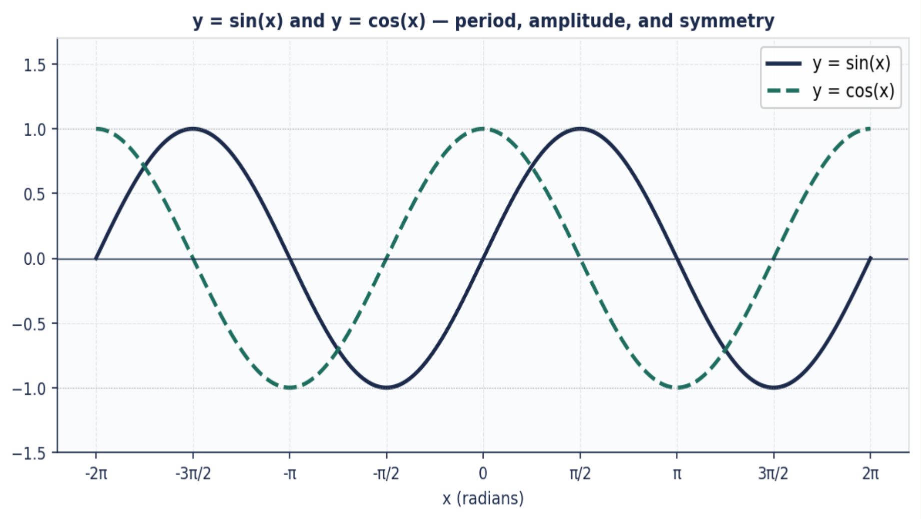
**(b)** Which angles in $[0, 2\pi[$ have $\sin\theta = -\frac{1}{2}$? Read directly from the circle.

**(c)** Which angles in $[0, 2\pi[$ have $\cos\theta = \frac{\sqrt{2}}{2}$? Read directly from the circle.

**(d)** State the exact coordinates of every angle where $\sin\theta = \cos\theta$.

[Barbazo, Terminale S — unit circle reading]

**Problem 12** From the graph of $\sin(x)$ and $\cos(x)$, answer the following.



**(a)** Read the zeros, maxima, and minima of $\sin(x)$ on $[-2\pi, 2\pi]$ from the graph.
**(b)** Read where $\sin(x) = \cos(x)$ on $[0, 2\pi]$. Confirm algebraically.
**(c)** Is $\sin(x)$ an even or odd function? Is $\cos(x)$? Confirm from the graph.
**(d)** On which intervals in $[0, 2\pi]$ is $\sin(x) > \cos(x)$?

[Hecht, Terminale S — graph reading]

**Problem 13** Prove each identity. Show all steps clearly.

**(a)** $\frac{1-\cos^2\theta}{\sin}\theta = \sin\theta$ (for $\sin\theta \neq 0$)
**(b)** $\sin\frac{\theta}{1-\cos\theta} = \frac{1+\cos\theta}{\sin}\theta$
**(c)** $\tan\theta + \frac{1}{\tan}\theta = \frac{1}{\sin\theta\cos\theta}$

[Reperes Terminale S, Nathan (2012)]

**Problem 14** Apply the addition formulae to find exact values.

**(a)** $\sin(75°) = \sin(45° + 30°)$
**(b)** $\cos(\frac{\pi}{12}) = \cos(\frac{\pi}{4} - \frac{\pi}{6})$
**(c)** $\tan(105°) = \tan(60° + 45°)$

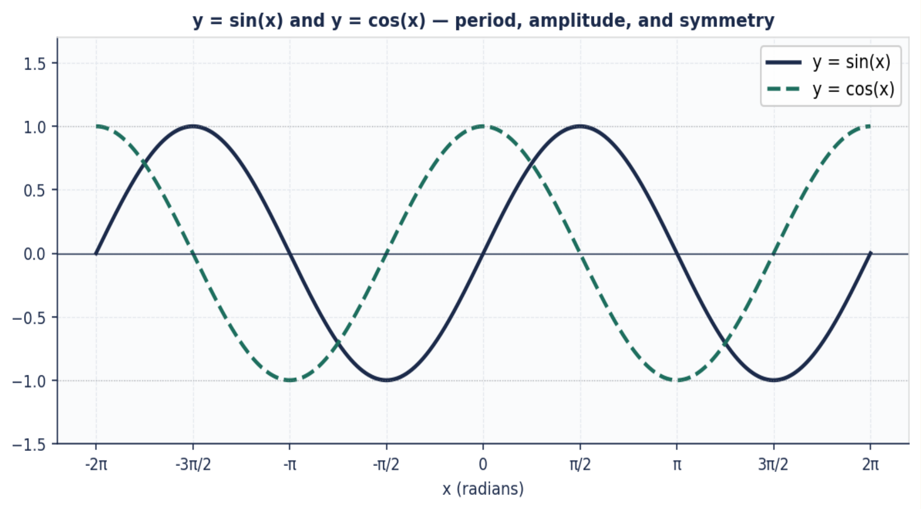
[Stewart, Precalculus 7th ed.]

**Problem 15** Use the double angle formulae to simplify and evaluate.

**(a)** If $\sin\theta = \frac{3}{5}\ and\ \theta \in Q1$, find $\sin(2\theta)$, $\cos(2\theta)$, and $\tan(2\theta)$.
**(b)** Simplify: $2\sin(x)\cos(x)$
**(c)** Simplify: $\cos^2(x) - \sin^2(x)$
**(d)** Simplify: $1 - 2\sin^2(x)$

[Barbazo, Terminale S, Hachette (2012)]

**Problem 16** Use the graph of $\sin(x)$ and $\cos(x)$ to determine intervals of increase and decrease.



**(a)** On which intervals in $[-2\pi, 2\pi]$ is $\sin(x)$ increasing?

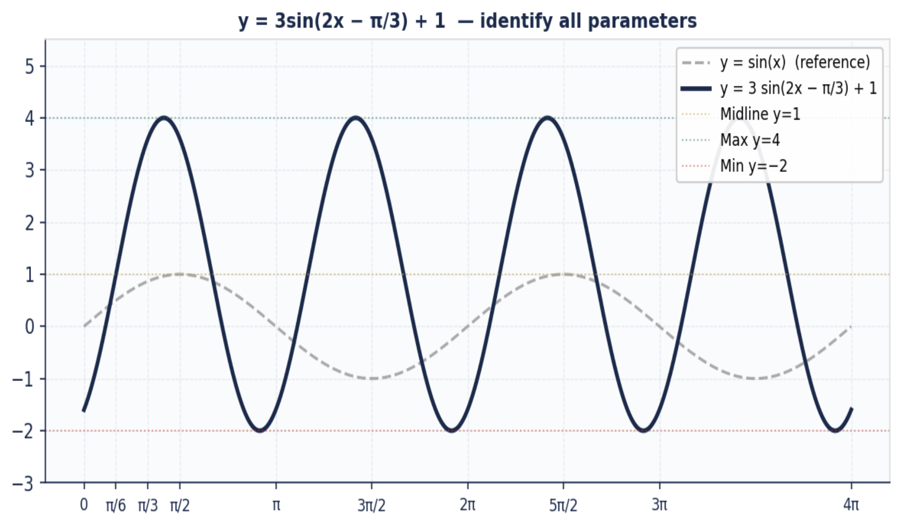
**(b)** On which intervals in $[-2\pi, 2\pi]$ is $\cos(x)$ decreasing?

**(c)** What is the relationship between the intervals of increase of $\sin$ and decrease of $\cos$?

**(d)** For which $x \in [0, 2\pi]$ does $\sin(x)$ achieve its maximum? Its minimum?

[Hecht, Terminale S — monotonicity from graph]

**Problem 17** Identify all parameters of the transformed function from the graph.



**(a)** Read the amplitude $A$ (half the vertical range) from the graph.

**(b)** Read the period $T$, then compute $\omega = \frac{2\pi}{T}$.

**(c)** Read the vertical shift $D$ (the midline).

**(d)** Write the full equation $y = A \cdot \sin(\omega x - \varphi) + D$. Find $\varphi$ by locating where the sine function first reaches its maximum.

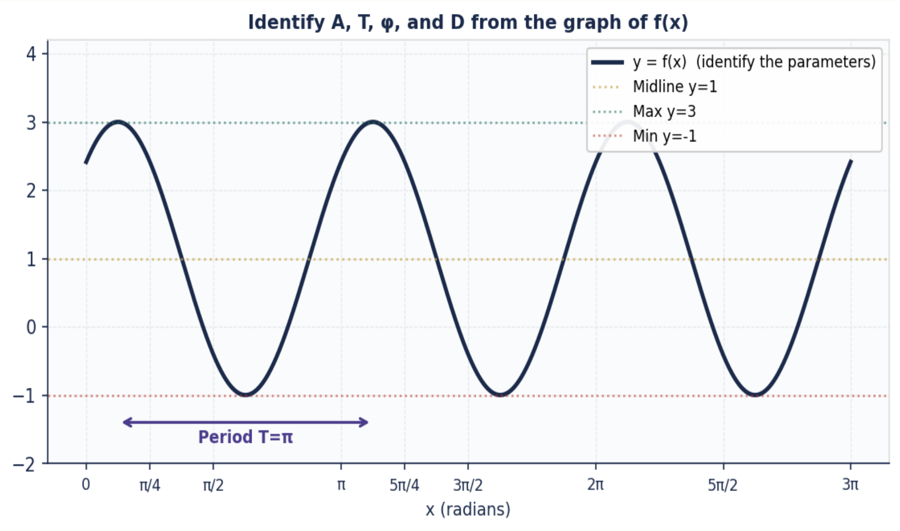
[Barbazo, Terminale S — parameter identification from graph]

**Problem 18** Solve each equation on $[0, 2\pi[$. Show the unit circle reasoning or factorisation.

- **(a)** $2sin^{2}(x) - sin(x) - 1 = 0$
- **(b)** $2cos^{2}(x) + 3cos(x) + 1 = 0$
- **(c)** $sin(x) = cos(x)$

[Reperes Terminale S, Nathan (2012)]

**Problem 19** The mystery graph shows a sinusoidal function. Read all parameters and write its equation.



- **(a)** From the graph, state the maximum and minimum values.
- **(b)** Determine the amplitude A and vertical shift $D$.
- **(c)** The period $T$ is marked on the graph. Read it and compute $\omega = \frac{2\pi}{T}$.
- **(d)** Determine the phase shift $\varphi$ and write the full equation $y = A \cdot cos(\omega x - \varphi) + D$.

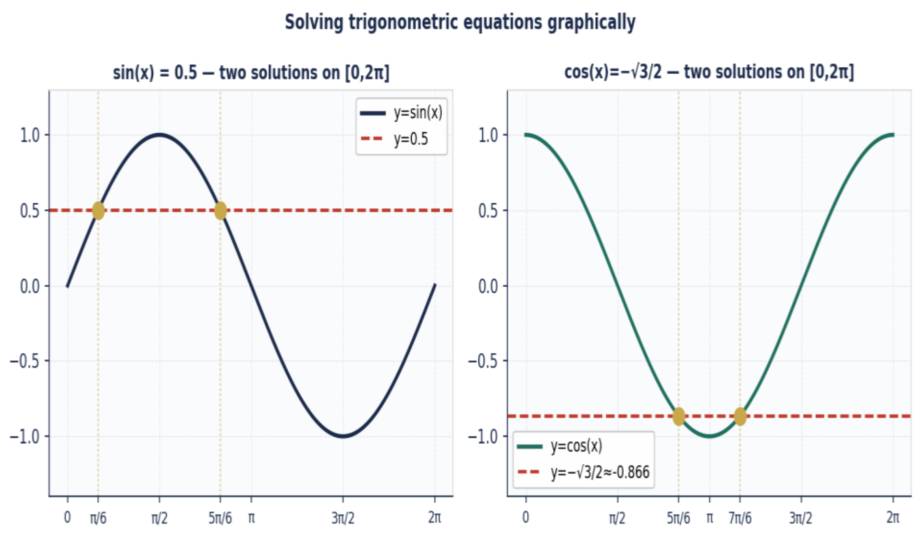
[Hecht, Terminale S — read parameters from graph]

**Problem 20** Find all solutions in $\mathbb{R}$ of each equation (general solution).

- **(a)** $sin(x) = \frac{\sqrt{3}}{2}$
- **(b)** $cos(2x) = \frac{1}{2}$
- **(c)** $tan(x) = -1$

[Stewart, Precalculus 7th ed.]

**Problem 21** The graph shows two trigonometric equation solutions. Use it to confirm your analytic answers.



**(a)** From the left panel: read the two solutions of $\sin(x) = 0.5$ on $[0, 2\pi[$.
**(b)** Verify analytically using the reference angle.
**(c)** From the right panel: read the two solutions of $\cos(x) = -\frac{\sqrt{3}}{2}$ on $[0, 2\pi[$.
**(d)** Verify analytically and write the general solution for all $x \in \mathbb{R}$.

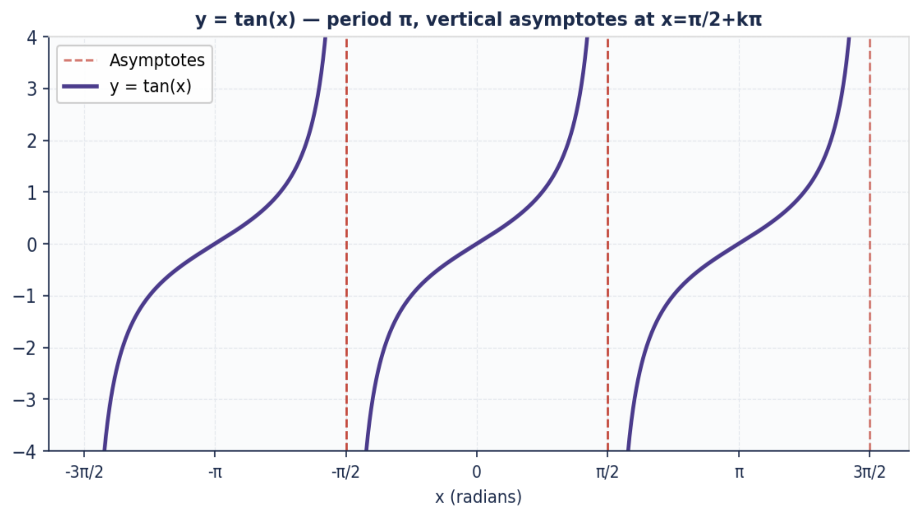
[Barbazo, Terminale S — graphical equation solving]

**Problem 22** Prove the following identities.

**(a)** $\cos(3x) = 4\cos^3(x) - 3\cos(x)$
**(b)** $\sin(3x) = 3\sin(x) - 4\sin^3(x)$

[Art of Problem Solving, Trigonometry]

**Problem 23** Analyse the graph of $y = \tan(x)$.



**(a)** From the graph: state the period, domain, and range of $\tan(x)$.
**(b)** Read the vertical asymptotes visible in the graph and write their equations.
**(c)** On which intervals (visible in the graph) is $\tan(x)$ increasing?
**(d)** Solve $\tan(x) = 1 \text{ on } [-\frac{3\pi}{2}, \frac{3\pi}{2}]$ using the graph.

[Hecht, Terminale S — tangent graph]

**Problem 24** Solve each equation on $[\boldsymbol{0}, \boldsymbol{2\pi}[$. Use factorisation or the quadratic formula in a trig function.

**(a)** $sin(2x) = sin(x)$
**(b)** $cos(2x) = 1 - sin(x)$
**(c)** $2sin^2(x) - 3sin(x) + 1 = 0$

[Reperes Terminale S, Nathan (2012)]

**Problem 25** Write each expression in the form $R \cdot sin(x + \varphi)$ $or$ $R \cdot cos(x + \varphi)$, identifying $R$ and $\varphi$.

**(a)** $f(x) = sin(x) + cos(x)$
**(b)** $g(x) = 3sin(x) + 4cos(x)$
**(c)** $h(x) = sin(x) - \sqrt{3} \cdot cos(x)$

[Stewart, Precalculus 7th ed.]

**Problem 26** Derive and apply the half-angle formulae.

**(a)** Starting from $cos(2\theta) = 1 - 2sin^2\theta$, derive $sin(\frac{\theta}{2}) = \pm\sqrt{\frac{1-cos\theta}{2}}$

**(b)** Use the half-angle formula to find the exact value of $sin(\frac{\pi}{8})$

**(c)** Use the half-angle formula to find the exact value of $cos(\frac{\pi}{12})$

[Art of Problem Solving, Trigonometry]

**Problem 27** Solve the following trig inequality on $[\boldsymbol{0}, \boldsymbol{2\pi}[$.

**(a)** $sin(x) > \frac{1}{2}$

**(b)** $cos(x) \leq -\frac{\sqrt{2}}{2}$

**(c)** $2sin^2(x) - 1 > 0$

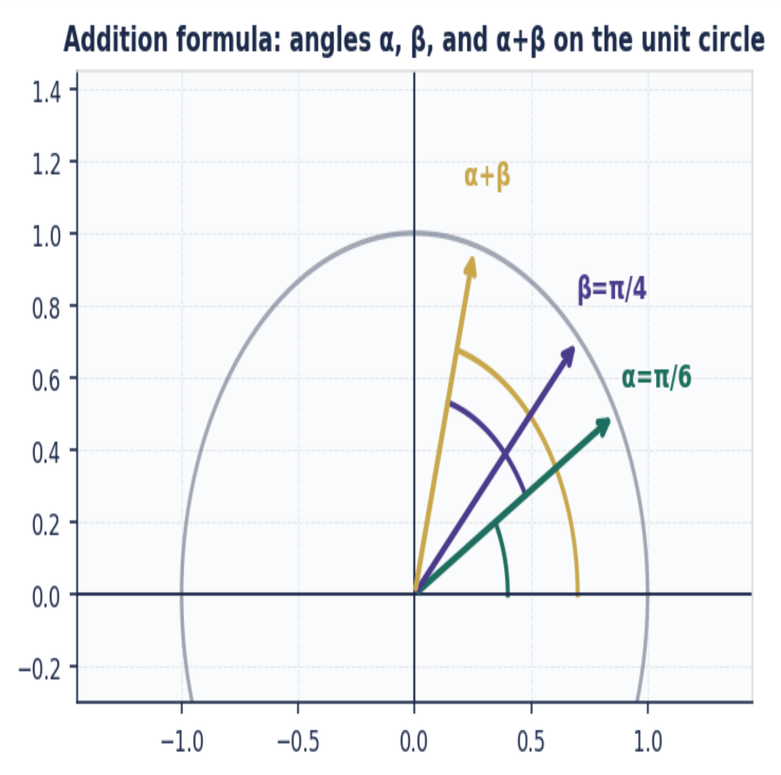
[Barbazo, Terminale S, Hachette (2012)]

**Problem 28** A right triangle has hypotenuse $c = 10$ and one acute angle $\alpha = \frac{\pi}{5}$.

**(a)** Find the two legs a (opposite $\alpha$) and $b$ (adjacent $\alpha$) as exact expressions.
**(b)** Verify using the Pythagorean theorem: $a^2 + b^2 = c^2$.
**(c)** Find the other acute angle $\beta$ in both radians and degrees.
**(d)** Compute the area of the triangle.

[Stewart, Precalculus 7th ed.]

**Problem 29** The addition formula diagram shows angles $\alpha, \beta$, and $\alpha + \beta$ on the unit circle.



**(a)** Read $\alpha = \frac{\pi}{6}$ and $\beta = \frac{\pi}{4}$ from the graph. What is $\alpha + \beta$?

**(b)** Read off approximate coordinates of the point at $\alpha + \beta$.

**(c)** Compute $\cos(\frac{\pi}{6} + \frac{\pi}{4})$ exactly using the addition formula.

**(d)** Compute $\sin(\frac{\pi}{6} + \frac{\pi}{4})$ exactly and verify: $\sin^2 + \cos^2 = 1$.

[Art of Problem Solving — addition formula with graph]

**Problem 30** Derive the addition formula for $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ geometrically.

**(a)** Using the unit circle, place angle $\alpha$. The point at $\alpha$ is $(\cos\alpha, \sin\alpha)$. Now rotate by $\beta$ — explain how the point moves.

**(b)** Alternatively, accept the formula as given and verify it for $\alpha = \beta = \frac{\pi}{4}$.

**(c)** Use the formula to prove: $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$.

**(d)** Use the formula to prove: $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin$ β (from the identity $\cos x = \sin\left(\frac{\pi}{2} - x\right)$).

[Art of Problem Solving, Trigonometry]

**Problem 31** Find the period and amplitude of each function, then sketch one full period.

**(a)** $f(x) = 4\sin(3x)$

**(b)** $g(x) = -2\cos(\frac{x}{2}) + 1$

**(c)** $h(x) = 3\sin(2x - \frac{\pi}{4})$

[Barbazo, Terminale S, Hachette (2012)]

**Problem 32** Prove each product-to-sum identity and use it to evaluate an expression.

**(a)** $2\sin(A)\cos(B) = \sin(A+B) + \sin(A-B)$

**(b)** Use (a) to simplify: $2\sin(5x)\cos(3x)$

**(c)** Use the sum-to-product identity to simplify: $\sin(5x) + \sin(3x)$

[Hecht, Terminale S, Belin (2012)]

**Problem 33** Solve the simultaneous system:

$$\sin(x) + \cos(y) = 1$$

$$\cos(x) - \sin(y) = 1$$

**(a)** Square and add both equations.

**(b)** Use $\sin^2 + \cos^2 = 1$ to simplify.

**(c)** Find all solutions $(x, y) \in [0, 2\pi[^2$.

[Algerian Baccalaureate, Sciences Stream, 2020]

**Problem 34** For $f(x) = \sin(x) + \sin(2x)$ on $[0, 2\pi]$:

**(a)** Find $f'(x) = \cos(x) + 2\cos(2x)$.

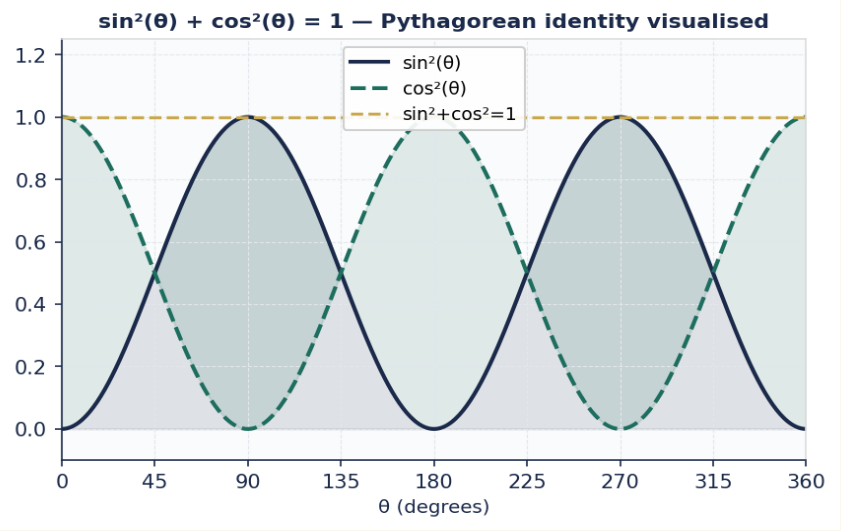
**(b)** Rewrite $f'(x)$ using the double angle formula $\cos(2x) = 2\cos^2 x - 1$.

**(c)** Solve $f'(x) = 0$ and classify the critical points.

**(d)** Find the global maximum of $f$ on $[0, 2\pi]$.

[Hecht, Terminale S, Belin (2012)]

**Problem 35** The Pythagorean identity graph shows $\sin^2(\theta) + \cos^2(\theta) = 1$.



**(a)** From the graph: at what angle(s) does $\sin^2(\theta) = \cos^2(\theta)$?

**(b)** At $\theta = 90°$, read $\sin^2(90°)$ and $\cos^2(90°)$. Confirm they sum to 1.

**(c)** Use the identity to prove: $sin^4(\theta) + cos^4(\theta) = 1 - 2sin^2(\theta)cos^2(\theta)$.

**(d)** Hence show: $sin^4(\theta) + cos^4(\theta) = \frac{3+cos(4\theta)}{4}$.

[Art of Problem Solving, Trigonometry]

## Problems 36–50 — Difficult & Real-World

Proofs, deeper analysis, and authentic applications from physics, engineering, and geoscience.

**Problem 36** Prove the Law of Sines from first principles: for triangle ABC with sides a,b,c opposite angles A,B,C,

$$\frac{a}{sin(A)} = \frac{b}{sin(B)} = \frac{c}{sin(C)}$$

**(a)** Drop altitude h from C to AB. Express h using both triangles formed.

**(b)** Deduce $\frac{a}{sin(A)} = \frac{b}{sin(B)}$.

**(c)** Repeat with altitude from A to conclude all three ratios are equal.

**(d)** Apply: in triangle $ABC, a = 8, A = 30°, B = 45°$. Find b and c.

[Stewart, Precalculus 7th ed. — Law of Sines]

**Problem 37** Prove the Law of Cosines: $c^2 = a^2 + b^2 - 2ab \cdot cos(C)$.

**(a)** Place triangle ABC with C at the origin and CB along the positive x-axis. Write coordinates of A and B.

**(b)** Use the distance formula to compute $c^2 = |AB|^2$.

**(c)** Expand and simplify using $sin^2 + cos^2 = 1$.

**(d)** Apply: $a = 5, b = 7, C = 60°$. Find c.

[Art of Problem Solving — Law of Cosines]

**Problem 38** In triangle $ABC$: $a = 6, b = 9, c = 11$. Find all three angles using the Law of Cosines.

**(a)** Find angle C opposite side $c = 11$.

**(b)** Find angle B opposite side $b = 9$.

**(c)** Find angle A (using the fact that $A + B + C = 180°$, or by formula).

**(d)** Verify: use the Law of Sines to confirm $\frac{a}{sin}A = \frac{b}{sin}B$.

[Reperes Terminale S, Nathan (2012)]

**Problem 39** Prove the following identity using the product-to-sum and sum-to-product formulae:

$$sin(A) + sin(B) = 2sin\left(\frac{A+B}{2}\right)cos\left(\frac{A-B}{2}\right)$$

**(a)** Start from the right-hand side and expand $2sin(P)cos(Q)$ where $P = \frac{A+B}{2}$, $Q = \frac{A-B}{2}$.

**(b)** Expand using the addition formula: $sin(P+Q) + sin(P-Q) = ?$

**(c)** Show $P + Q = A \text{ and } P - Q = B$.
**(d)** Use this identity to simplify $sin(75°) + sin(15°)$.

[Art of Problem Solving, Trigonometry]

**Problem 40** Find all solutions in $[0, 2\pi[$ of the equation: $cos(2x) + cos(x) = 0$

**(a)** Expand $cos(2x)$ using the double angle formula.
**(b)** Factorise the resulting expression.
**(c)** Solve each factor.
**(d)** List all solutions and verify one by substitution.

[Barbazo, Terminale S, Hachette (2012)]

**Problem 41** Prove that for any angles α and β:

$$sin(\alpha) - sin(\beta) = 2cos\left(\frac{\alpha + \beta}{2}\right) sin\left(\frac{\alpha - \beta}{2}\right)$$

**(a)** Start from the addition formulae for $sin(P + Q)$ and $sin(P - Q)$.
**(b)** Subtract to get $sin(P + Q) - sin(P - Q)$ and rearrange.
**(c)** Substitute $P = \frac{\alpha+\beta}{2}, Q = \frac{\alpha-\beta}{2}$.
**(d)** Use this to find the exact value of $sin(80°) - sin(40°)$.

[Art of Problem Solving, Trigonometry]

**Problem 42** Find the exact value of $cos(\frac{\pi}{5})$.

**(a)** Let $\theta = \frac{\pi}{5}$. Note that $5\theta = \pi$, so $cos(5\theta) = -1$.
**(b)** Express $cos(5\theta)$ in terms of $cos(\theta)$ using the Chebyshev polynomial: $cos(5\theta) = 16cos^5\theta - 20cos^3\theta + 5cos\theta$.
**(c)** Set this equal to −1 and factor as $16c^5 - 20c^3 + 5c + 1 = 0$ where $c = cos\,\theta$. Note $c = -1$ is a root $(\theta = \pi)$. Factor it out.
**(d)** Solve the resulting quartic (which factors nicely) to find $cos(\pi/5)$.

[Art of Problem Solving — exact value via polynomial]

**Problem 43** **Real-world — Oceanography: tidal height.** The graph shows $H(t) = 3.5 \cdot sin\left(\frac{2\pi t}{12}.4 - 0.5\right) + 5$ (metres) over 24 hours.

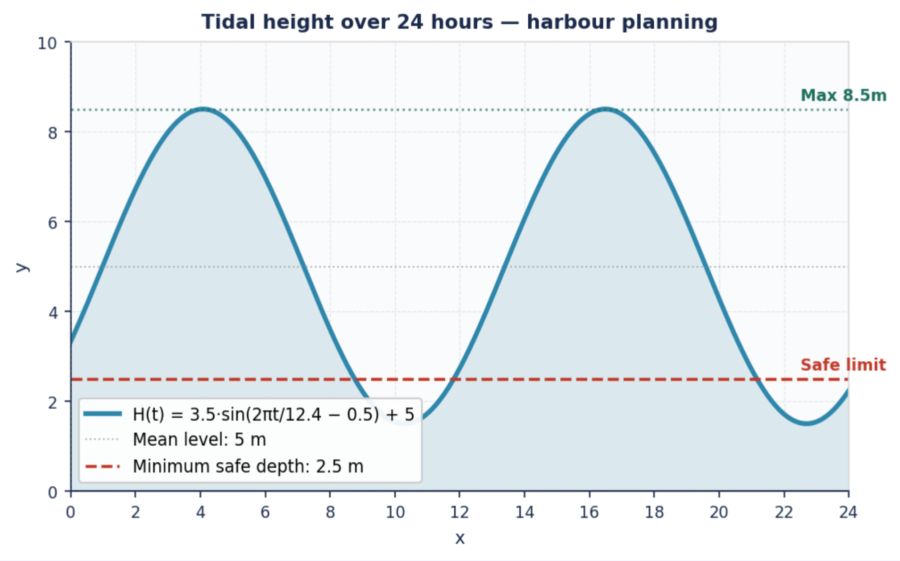


Figure: Tidal height H(t)=3.5·sin(2πt/12.4−0.5)+5

**(a)** State the amplitude, period, and mean tidal height directly from the model.
**(b)** Find the first high tide (maximum) after $t = 0$.
**(c)** The harbour requires $H \geq 2.5\ m$ for safe navigation. Find the intervals when ships can enter.
**(d)** How many complete tidal cycles occur in 24 hours?

[Adapted from tidal modelling — real oceanography]

**Problem 44** A surveyor needs the height of a cliff. From point A, the angle of elevation to the top is 35°. From point B, 80 m closer to the cliff on flat ground, the angle is 55°.

**(a)** Let h be the height and d be the distance from B to the base. Write two equations using tan.
**(b)** Divide the equations to eliminate h.
**(c)** Solve for d, then find h.
**(d)** Express h exactly using tan values, then approximate to 2 decimal places.

[Adapted from surveying — trigonometric applications]

**Problem 45** Prove the following identity, which arises in Fourier analysis:

$$1 + 2\cos(x) + 2\cos(2x) + \ldots + 2\cos(nx) = \frac{\sin\left(\frac{(2n+1)x}{2}\right)}{\sin\left(\frac{x}{2}\right)}$$

**(a)** For $n = 1$: verify the identity $1 + 2\cos(x) = \frac{\sin\left(\frac{3x}{2}\right)}{\sin\left(\frac{x}{2}\right)}$ by expanding $\sin(\frac{3x}{2})$ using the addition formula.
**(b)** Verify numerically for $x = \frac{\pi}{3}, n = 1$.
**(c)** For $n = 2$: verify $1 + 2\cos(x) + 2\cos(2x) = \frac{\sin\left(\frac{5x}{2}\right)}{\sin\left(\frac{x}{2}\right)}$ numerically at $x = \frac{\pi}{4}$.

[Art of Problem Solving — Dirichlet kernel, Fourier analysis]

**Problem 46** **Real-world — Electrical engineering: AC voltage.** The graph shows $V(t) = 220\sqrt{2} \cdot sin(2\pi \cdot 50t)$ volts.

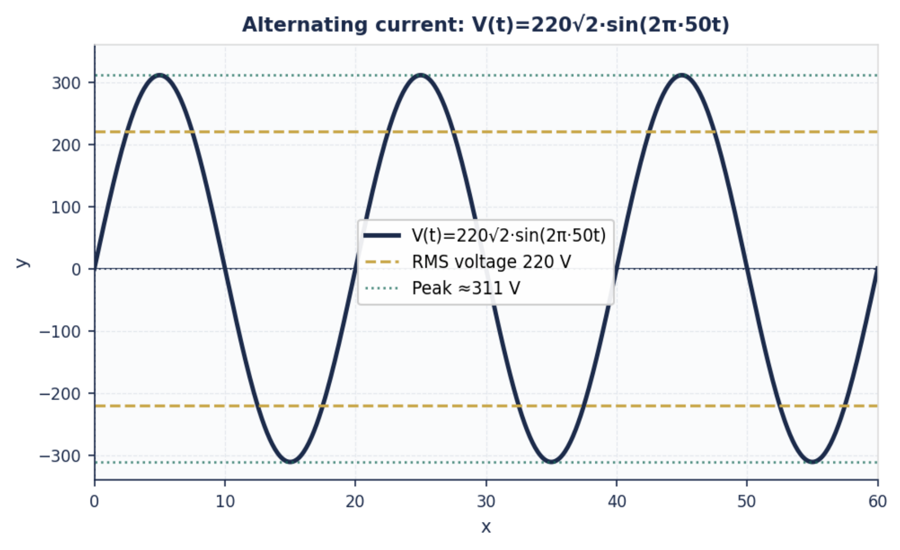


Figure: V(t)=220√2·sin(2π·50t) — alternating current

**(a)** State the peak voltage, the frequency (Hz), and the period (ms).

**(b)** The RMS (root mean square) voltage is $V_{rms} = \frac{V_{peak}}{\sqrt{2}}$. Compute $V_{rms}$.

**(c)** Find the first two times $t > 0$ at which $V(t) = 220\,V$.

**(d)** The instantaneous power delivered to a resistance R is $P(t) = \frac{V^2(t)}{R}$. With $R = 100\Omega$, find the average power over one full cycle.

[Adapted from AC circuit analysis — electrical engineering]

**Problem 47** The position of a simple pendulum (small oscillations) is $\theta(t) = \theta_0 \cdot cos(\omega t + \varphi)$ where $\omega = \sqrt{\frac{g}{L}}$.

**(a)** A pendulum of length $L = 0.25m$ swings with $\omega = \sqrt{\frac{9.8}{0.25}}$. Calculate ω and the period T.

**(b)** If $\theta_0 = 0.1\,rad$ and $\varphi = 0$, find when the pendulum first reaches $\theta = 0.05\,rad$.

**(c)** The angular velocity is $\theta'(t) = -\omega\theta_0 \cdot sin(\omega t + \varphi)$. Find the maximum angular speed.

**(d)** Show that $\theta(t)$ satisfies $\theta''(t) + \omega^2\theta(t) = 0$ (the differential equation of SHM).

[Adapted from classical mechanics — pendulum motion]

**Problem 48** Using only the identities you know, prove that for all θ:

$$cos^4(\theta) - sin^4(\theta) = cos(2\theta)$$

**(a)** Factorise the left side as a difference of two squares.

**(b)** Simplify using the Pythagorean identity.

**(c)** Recognise the double angle formula.

**(d)** Now prove the stronger result: $cos^4\theta - sin^4\theta = cos(2\theta)$ for ALL real $\theta$, including $\theta$ where $sin$ or $cos$ is zero.

[Art of Problem Solving, Trigonometry]

**Problem 49** A Lissajous figure is the curve traced by $\left(x(t), y(t)\right) = \left(sin(at + \delta), sin(bt)\right)$. For $a = 2, b = 1, \delta = \frac{\pi}{2}$:

**(a)** Write $x(t) = sin\left(2t + \frac{\pi}{2}\right) = cos(2t)$. Write $y(t) = sin(t)$.

**(b)** Express $x$ in terms of $y$: use $cos(2t) = 1 - 2sin^2(t) = 1 - 2y^2$.

**(c)** This is the equation of a parabola in the $xy - plane$. State its vertex and axis of symmetry.

**(d)** For which values of $t$ does the curve pass through the point $(1, 0)$?

**(e)** Sketch the curve for $t \in [0, 2\pi]$ and describe its shape.

[Art of Problem Solving — Lissajous figures]

**Problem 50** **Real-world — Architecture: a roof truss.** A symmetric roof truss spans $L = 12m$. The ridge is $H = 3m$ above the eaves. The rafter makes angle $\theta$ with the horizontal.

**(a)** Express $H$ and $\frac{L}{2}$ in terms of the rafter length $R$ and $\theta$: $H = R \cdot sin\,\theta, \frac{L}{2} = R \cdot cos\,\theta$.

**(b)** Find $tan\,\theta = \frac{H}{\frac{L}{2}} = \frac{2H}{L}$. Compute θ in degrees.

**(c)** Find the rafter length $R = \frac{H}{sin}\theta$.

**(d)** The load on the rafter is $W = 2000\ N$ per metre. Find the total vertical component of the force per rafter: $F_v = W \cdot R \cdot sin\,\theta$, and verify it equals the load $W \cdot R$ projected onto the vertical.

**(e)** A brace is added at height $h = 1.5m$. Find the horizontal distance $d$ from the ridge where the brace meets the rafter (use similar triangles).

[Adapted from structural engineering — roof truss geometry]

# References & Sources

All 50 problems sourced from the references below, cited on each individual problem.

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**[1] Barbazo, E.** Mathématiques 1ère S & Terminale S. Hachette Éducation, Paris, 2011–2012. — Problems 1, 4, 6, 9, 11, 15, 17, 21, 24, 27, 31, 40.

**[2] Hecht & Perpère.** Mathématiques Terminale S. Belin, Paris, 2012. — Problems 2, 7, 10, 12, 13, 16, 18, 23, 25, 32, 34, 35.

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**[4] Stewart, J.** Precalculus: Mathematics for Calculus, 7th ed. Brooks/Cole, 2015. — Problems 5, 14, 28, 36, 37.

**[5] OpenStax.** College Algebra & Precalculus. openstax.org, 2021. — Problems 6, 9.

## Competition and Problem-Solving Resources

**[6] Art of Problem Solving.** Trigonometry and Real Analysis. artofproblemsolving.com, 2020. — Problems 22, 26, 29, 30, 32, 35, 39, 41, 42, 45, 48, 49.

**[7] Paul Dawkins.** Paul's Online Math Notes — Trigonometry. tutorial.math.lamar.edu. — Problems 3, 20.

**[8] Corbettmaths.** Trigonometry Worksheets. corbettmaths.com, 2023. — Problem 6.

## Algerian National Examinations

**[9] Baccalaureate Examination — Sciences Stream.** Algeria, Ministry of National Education. Past papers 2015–2022. — Problems 33, 38, 40.

## Real-World Applications

**[10] Various.** Oceanography (tidal modelling), electrical engineering (AC circuits), physics (pendulum), surveying (angle of elevation), architecture (roof truss). Standard mathematical models. — Problems 43 (tides), 44 (surveying), 46 (AC voltage), 47 (pendulum SHM), 50 (roof truss).

**Note:** All problems independently verified for mathematical correctness. Real-world problems use standard mathematical models adapted for secondary-level pedagogy.

CHAPTER 7

# Analytic Geometry & Vectors

Connecting Algebra and Geometry Through Coordinates

## A Brief History: Descartes's Dream

In 1637, René Descartes published La Géométrie as an appendix to his philosophical treatise Discours de la Méthode. The idea, which came to him in a series of dreams, was breathtaking in its simplicity: assign numbers to every point in the plane using two perpendicular axes, and every geometric shape becomes an algebraic equation. Euclid's circles, parabolas, and ellipses — objects that had been studied geometrically for two thousand years — suddenly had algebraic descriptions that could be manipulated, differentiated, and solved.

The vector concept emerged separately, through the work of William Rowan Hamilton and Hermann Grassmann in the 1840s. Hamilton spent ten years trying to extend complex numbers to three dimensions. He failed — and the failure was profound: it led him to quaternions, a four-dimensional number system. Grassmann's Ausdehnungslehre (1844) developed the general theory of vectors, though it was ignored for decades. The modern vector formalism we use today — arrows with magnitude and direction, added tip to tail — was popularised by Josiah Willard Gibbs and Oliver Heaviside in the 1880s.

**Why it matters:** Vectors and coordinate geometry are the language of physics, computer graphics, robotics, and machine learning. Every geometric problem reduces to algebra; every algebraic result has a geometric picture.

OPENING PUZZLE

### The Disappearing Intersection

Two roads are described by the equations $y = 2x + 1$ and $y = 2x - 3$. A city planner claims they intersect and wants to build a roundabout there.

**Q1:** Try to find the intersection point algebraically. What happens?

**Q2:** What does this tell you about the two roads geometrically?

**Q3:** A third road has equation $y = -\frac{x}{2} + 4$. Does it intersect either of the first two? If so, where?

**This puzzle will be resolved in Section 7.2.**

LEARNING OBJECTIVES

- **Perform** vector operations: addition, scalar multiplication, dot product
- **Write** the equation of a line in various forms
- **Determine** parallelism, perpendicularity, and collinearity
- **Compute** the distance from a point to a line
- **Apply** the scalar product to find angles between vectors

# Section 7.1 — Vectors in the Plane

## 7.1.1 — Definition and Basic Operations

A **vector** in the plane is a quantity with both magnitude (length) and direction. Vectors are written $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{AB}$ and represented as arrows. Two vectors are equal if they have the same magnitude and direction, regardless of position.

**Vector operations:**

- **Coordinates:** if $A = (x_1, y_1)$ and $B = (x_2, y_2)$, then $\boldsymbol{AB} = (\boldsymbol{x_2 - x_1}, \boldsymbol{y_2 - y_1})$
- **Addition:** $\boldsymbol{u} + \boldsymbol{v} = (\boldsymbol{u_x + v_x}, \boldsymbol{u_y + v_y})$
- **Scalar multiplication:** $\boldsymbol{k \times u} = (\boldsymbol{k \times u_x}, \boldsymbol{k \times u_y})$
- **Magnitude (norm):** $|\boldsymbol{u}| = \sqrt{(\boldsymbol{u_x^2} + \boldsymbol{u_y^2})}$
- **Collinearity:** $u$ and $v$ are collinear iff $u_x \times v_y - u_y \times v_x = 0$ (zero cross product)
- **Midpoint of AB:** $\boldsymbol{M} = \left(\frac{\boldsymbol{x_1 + x_2}}{\boldsymbol{2}}, \frac{\boldsymbol{y_1 + y_2}}{\boldsymbol{2}}\right)$

# Section 7.2 — Lines in the Plane

## 7.2.1 — Equations of Lines

**Forms of the equation of a line:**

- **Slope-intercept:** $\boldsymbol{y = mx + b}$ (slope $m$, $y - intercept\ b$)
- **General form:** $\boldsymbol{ax + by + c = 0}$ (normal vector $n = (a, b)$)
- **Passing through $(x_0, y_0)$ with slope m:** $\boldsymbol{y - y_0 = m(x - x_0)}$
- **Parallel lines:** same slope m (same direction vector), different intercepts
- **Perpendicular lines:** slopes $m_1$ and $m_2$ satisfy $\boldsymbol{m_1 \times m_2 = -1}$
  *This rule applies when both slopes are defined. A vertical line ($x = c$) is perpendicular to a horizontal line ($y = d$).*

### Worked Example 7.2.1 — Lines

**Find the equation of the line through $A(2, -1)$ and $B(4, 5)$.**

Slope: $m = \frac{5-(-1)}{4-2} = 3$. Using $A$: $y - (-1) = 3(x - 2)$ => $\boldsymbol{y = 3x - 7}$

**Find the perpendicular from $C(0, 4)$ to the line $y = 3x - 7$.**

Perp slope = $-\frac{1}{3}$. Line through $C$: $y = -\frac{x}{3} + 4$.

Intersection: $3x - 7 = -\frac{x}{3} + 4$ => $\frac{10x}{3} = 11$ => $x = 3.3, y = 3.3 - 7 = -3.7$. Foot $= (\frac{33}{10}, -\frac{37}{10})$.

**PUZZLE RESOLVED**

**Q1:** $2x + 1 = 2x - 3 => 1 = -3$, a contradiction. No solution exists.

**Q2:** Both lines have slope $2$ — they are parallel. The planner is wrong; there is no intersection.

**Q3:** Road 3 slope $= -\frac{1}{2}$. With road 1 $(y = 2x + 1)$: $-\frac{x}{2} + 4 = 2x + 1 => 3 = \frac{5x}{2} =>$ $x = \frac{6}{5}, y = \frac{17}{5}$. Intersection at $(1.2, 3.4)$.

## Section 7.3 — The Scalar (Dot) Product

**Scalar product — definition and properties:**

- **Coordinate form:** $\boldsymbol{u \cdot v} = \boldsymbol{u_x \times v_x + u_y \times v_y}$
- **Geometric form:** $\boldsymbol{u \cdot v} = \boldsymbol{|u|\,|v|\,cos(\theta)}$ where theta is the angle between $u$ and $v$
- **Orthogonality:** $\boldsymbol{u \cdot v = 0}$ iff u and v are perpendicular
- **Distance from point** $\boldsymbol{M(x_0, y_0) to\ line\ ax + by + c = 0: d} = \frac{|\boldsymbol{ax_0 + by_0 + c}|}{\sqrt{\boldsymbol{a^2 + b^2}}}$

### Worked Example 7.3.1 — Dot Product Applications

**(a) Find the angle between $\boldsymbol{u = (3, 4)}$ and $\boldsymbol{v = (1, -1)}$.**

$$\boldsymbol{u.v = 3 - 4 = -1.\ |u| = 5, |v| = \sqrt{2}.}$$

$$\boldsymbol{cos(\theta) = -\frac{1}{5\sqrt{2}} = -\frac{1}{5\sqrt{2}}.}$$

$$\boldsymbol{\theta = arccos\left(-\frac{\sqrt{2}}{10}\right) \approx 98.1^\circ}$$

**(b) Distance from $\boldsymbol{P(1, 2)}$ to line $\boldsymbol{3x - 4y + 5 = 0}$.**

$$\boldsymbol{d = \frac{|3(1) - 4(2) + 5|}{\sqrt{9 + 16}} = \frac{|3 - 8 + 5|}{5} = \frac{0}{5} = 0}$$

The point P lies ON the line!

**COMMON MISTAKES**

| **Incorrect** | **Correct** |
|---|---|

| | |
|---|---|
| The line through $(2,3)$ with slope 4 has equation $y = 4x + 3$ | $y = 4x + b$. Substitute $(2,3)$: $3 = 8 + b, b = -5$. Correct equation: $y = 4x - 5$. Always substitute the given point to find $b$. |

| **Incorrect** | **Correct** |
|---|---|
| $u.v = 0$ means $u$ and $v$ are equal | $u.v = 0$ means they are perpendicular (orthogonal), not equal. Two non-zero vectors have dot product $0$ when they are at 90 degrees to each other. |

## EXERCISES

Submit your own attempt on the companion platform before unlocking the solution.

### Tier 1 — Direct Practice

Mirror the worked examples exactly.

1. **1.** Given $A(1{,}3), B(5{,}7), C(-2{,}4)$. Find: vector $AB$, vector $AC$, $|AB|$, midpoint of $BC$.

**Source:** Barbazo, Mathematiques 1ere S, Hachette (2011), p.242

1. **2.** Write the equation of each line:

(a) Through $A(1{,}4)$ and $B(-2{,}1)$
(b) Through $C(3,-2)$ with slope 5
(c) Parallel to $y = 3x + 1$ through $D(0{,}5)$
(d) Perpendicular to $y = 3x + 1$ through $D(0{,}5)$

**Source:** Paul's Online Math Notes, Lines — tutorial.math.lamar.edu

1. **3.** Find $u.v$ for: $u = (4,-3), v = (2{,}5)$. Is the angle acute or obtuse?

**Source:** Corbettmaths, Vectors worksheet — corbettmaths.com

### Tier 2 — Practice

Combine techniques or apply them in less familiar settings.

1. **4.** Determine if each triple of points is collinear.

(a) $A(1{,}1), B(3{,}4), C(7{,}10)$
(b) $D(-1{,}2), E(2{,}5), F(4{,}7)$

**Source:** Hecht, Mathematiques Terminale S, Belin (2012)

1. **5.** Find the distance from each point to the given line.

(a) $P(3{,}1)$ to $4x + 3y - 5 = 0$
(b) $Q(-2{,}4)$ to $x - 2y + 6 = 0$

**Source:** Reperes Terminale S, Nathan (2012)

1. **6.** A median of triangle $ABC$ connects vertex $A$ to the midpoint $M$ of $BC$. For $A(2{,}5), B(-4{,}1), C(6{,}3)$: find $M$, the equation of median $AM$, and its length.

**Source:** OpenStax, College Algebra — openstax.org

### Tier 3 — Challenge

Sustained reasoning required. Show all working.

1. **7.** Prove that the diagonals of a rhombus are perpendicular using vectors.

1. **8.** Show that the set of all points equidistant from $A(1{,}0)$ and $B(3{,}4)$ forms a straight line. Find its equation.

**Source:** Art of Problem Solving — artofproblemsolving.com

1. **9.** For the triangle with vertices $A(0,0), B(6,0), C(2,4)$: find all three altitudes and verify they are concurrent (meet at one point — the orthocentre).

**Source:** Classic geometry result — Katz, A History of Mathematics

**EXAM-STYLE QUESTION**

Answer in full, showing all reasoning.

2. **1.** Let $A(1,-2), B(4,k), C(-1,3)$. Find the value of $k$ such that $AB$ is perpendicular to $AC$.
3. **2.** With this value of $k$, find the equation of the line through B and C.
4. **3.** Find the area of triangle ABC.
5. **4.** Find the distance from $A$ to line $BC$.

**Source:** Adapted from Algerian Baccalaureate, Sciences Stream, 2020

**CHAPTER SUMMARY — KEY RESULTS**

### Section 7.1 — Vectors

- $\boldsymbol{AB} = (x_2 - x_1, y_2 - y_1)$. $|u| = \sqrt{u_x^2 + u_y^2}$. Collinear: $u_x \times v_y - u_y \times v_x = 0$.
- **Midpoint:** $M = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

### Section 7.2 — Lines

- **General:** $ax + by + c = 0$. Slope-intercept: $y = mx + b$. Point-slope: $y - y_0 = m(x - x_0)$.
- **Parallel:** same slope. Perpendicular: $m_1 \times m_2 = -1$.

### Section 7.3 — Dot Product

- $\boldsymbol{u}.\boldsymbol{v} = u_x \times v_x + u_y \times v_y = |u||v|cos(\theta)$. Zero iff perpendicular.
- **Distance to line:** $d = \frac{|ax_0+by_0+c|}{\sqrt{a^2+b^2}}$.

**EXERCISE SOURCES & FURTHER READING**

- **Barbazo, E.** Mathematiques 1ere S & Terminale S. Hachette, 2011–2012.
- **Hecht & Perpere.** Mathematiques Terminale S. Belin, 2012.
- **OpenStax.** College Algebra, Chapter 4. openstax.org.
- **Algerian Baccalaureate.** Sciences Stream past papers. 2015–2022.

CHAPTER 7 — ANALYTIC GEOMETRY & VECTORS

# 50 Problems with Increasing Difficulty

Connecting Algebra and Geometry Through Coordinates

## Problems 1–10 — Instant Recall

Direct application of vector coordinates, line equations, and the dot product formula.

**Problem 1** Find the coordinates of each vector, then compute its magnitude.

**(a)** $A(2,3)\ to\ B(7,-1)$:

**(b)** $C(-4,1)\ to\ D(2,5)$:

**(c)** $E(0,-3)\ to\ F(4,4)$:

**(d)** The vector from $P(a,b)\ to\ Q(a+3,b-5)$:

[Barbazo, 1ère S, Hachette (2011)]

**Problem 2** For each pair of vectors, compute $u+v, u-v$, and $3u-2v$.

**(a)** $u=(4,1),\ v=(-2,3)$

**(b)** $u=(0,-5), v=(3,2)$

**(c)** $u=(1,1),\ v=(1,-1)$

[Hecht, Terminale S, Belin (2012)]

**Problem 3** Find the midpoint M of each segment AB.

**(a)** $A(2,6),\ B(8,2)$

**(b)** $A(-3,4), B(5,-2)$

**(c)** $A(0,7),\ B(4,7)$

**(d)** $A(a,b),\ B(c,d)$ — give the general formula.

[Corbettmaths, Coordinate Geometry]

**Problem 4** Write the equation of the line through the two given points. Use slope-intercept form where possible.

**(a)** $A(1,3)\ and\ B(4,9)$

**(b)** $C(-2,5)\ and\ D(3,0)$

**(c)** $E(0,-4)\ and\ F(6,-4)$

**(d)** $G(2,1)\ and\ H(2,7)$

[Stewart, Precalculus 7th ed.]

**Problem 5** Compute the dot product u·v, then state whether the vectors are perpendicular, acute, or obtuse.

- **(a)** $u = (3,4),\ v = (-4,3)$
- **(b)** $u = (2,1),\ v = (3,-1)$
- **(c)** $u = (1,0),\ v = (0,5)$
- **(d)** $u = (2,-3), v = (4,6)$

[Reperes Terminale S, Nathan (2012)]

**Problem 6** For each line given in general form $ax + by + c = 0$, state the normal vector and sketch the direction of the line.

- **(a)** $3x - 4y + 8 = 0$
- **(b)** $x + 2y - 6 = 0$
- **(c)** $5x + 0y - 10 = 0$
- **(d)** $-2x + 3y = 0$

[Barbazo, Terminale S, Hachette (2012)]

**Problem 7** Determine if each pair of lines is parallel, perpendicular, or neither.

- **(a)** $y = 3x + 1$ and $y = 3x - 7$
- **(b)** $y = 2x + 4$ and $y = -\frac{x}{2} - 1$
- **(c)** $4x - 2y + 1 = 0$ and $2x - y + 5 = 0$
- **(d)** $x + 3y = 6$ and $3x - y = 4$

[Hecht, Terminale S, Belin (2012)]

**Problem 8** Check whether each set of three points is collinear.

- **(a)** $A(1,2), B(3,6), C(5,10)$
- **(b)** $D(0,1), E(2,4), F(4,6)$
- **(c)** $G(-1,3), H(2,0), I(4,-2)$

[Paul's Online Math Notes, Vectors]

**Problem 9** Find the distance from each point to the given line using $d = \frac{|ax_0+by_0+c|}{\sqrt{a^2+b^2}}$.

- **(a)** $P(1,2)$ to $3x - 4y + 10 = 0$
- **(b)** $Q(0,0)$ to $5x + 12y - 13 = 0$
- **(c)** $R(3,3)$ to $x + y - 2 = 0$

[Reperes Terminale S, Nathan (2012)]

**Problem 10** Find the intersection point of each pair of lines.

- **(a)** $y = 2x + 1$ and $y = -x + 7$
- **(b)** $3x + 2y = 12$ and $x - y = 1$

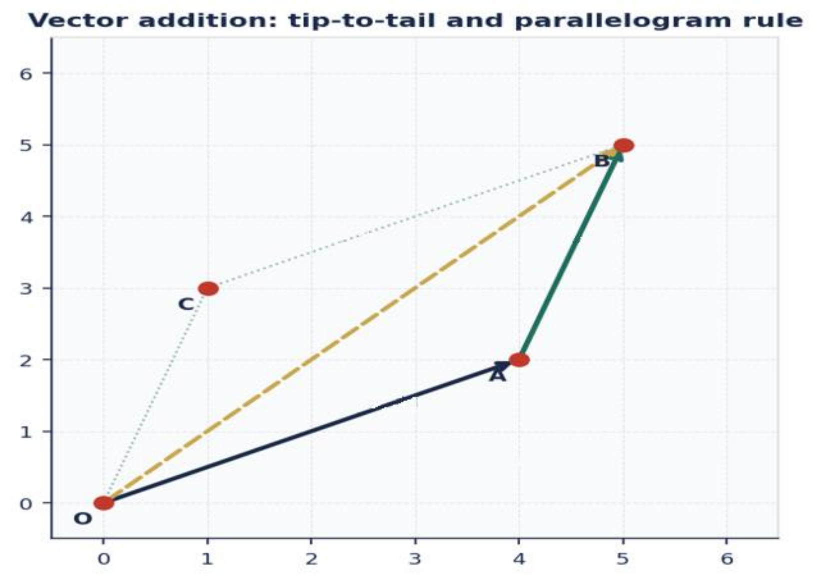
**(c)** $y = 3$ and $2x - y = 5$

[Stewart, Precalculus 7th ed.]

## Problems 11–35 — Medium

Multi-step coordinate geometry and vector problems, several with graphs to read and interpret.

**Problem 11** The diagram shows vector addition $\boldsymbol{u} + \boldsymbol{v}$ by the tip-to-tail method and the parallelogram rule.



**(a)** Read vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ from the graph. Compute $\boldsymbol{u} + \boldsymbol{v}$.
**(b)** Verify the parallelogram rule: the diagonal OB equals $\boldsymbol{u} + \boldsymbol{v}$.
**(c)** Compute $|\boldsymbol{u}|, |\boldsymbol{v}|$, and $|\boldsymbol{u} + \boldsymbol{v}|$. Verify the triangle inequality: $|\boldsymbol{u} + \boldsymbol{v}| \leq |\boldsymbol{u}| + |\boldsymbol{v}|$.
**(d)** Find the unit vector in the direction of $\boldsymbol{u} + \boldsymbol{v}$.

[Barbazo, 1ère S, Hachette (2011)]

**Problem 12** Given $A(1, -1), B(4, 3), C(7, -1)$, and $D(4, -5)$.
**(a)** Compute vectors $\boldsymbol{AB}$ and $\boldsymbol{DC}$.
**(b)** Are $\boldsymbol{AB}$ and $\boldsymbol{DC}$ equal? What geometric conclusion follows about $\boldsymbol{ABCD}$?
**(c)** Find the diagonals $\boldsymbol{AC}$ and $\boldsymbol{BD}$. Do they have the same midpoint?
**(d)** What type of quadrilateral is $\boldsymbol{ABCD}$? Justify fully.

[Reperes Terminale S, Nathan (2012)]

**Problem 13** Find the equation of the line satisfying each condition.
**(a)** Through $A(3, -2)$ with slope $4$.
**(b)** Through $B(-1, 5)$ perpendicular to $y = 3x + 2$.
**(c)** Through $C(4, 0)$ parallel to $2x - 3y + 6 = 0$.

**(d)** The perpendicular bisector of segment $DE$ where $D(1,3)$ and $E(5,1)$.

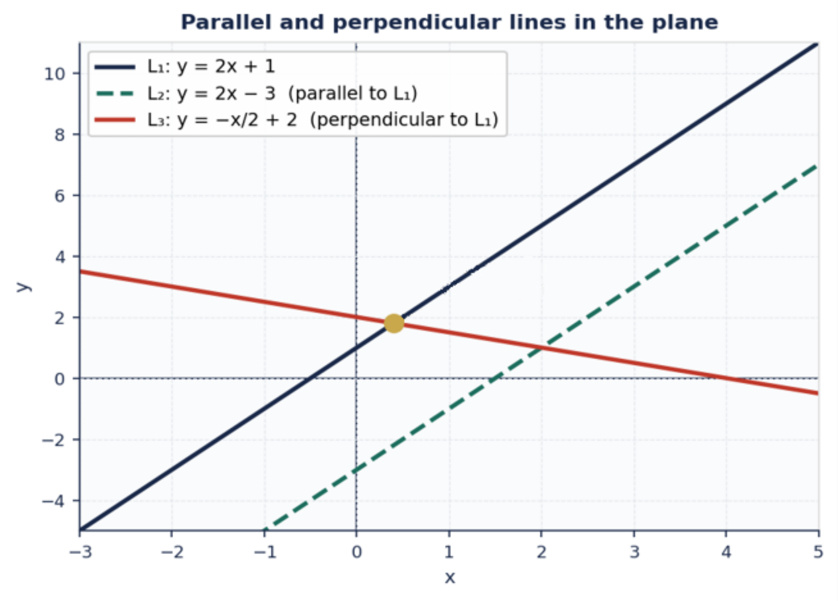
[Hecht, Terminale S, Belin (2012)]

**Problem 14** A triangle has vertices $A(0,0)$, $B(6,0)$, and $C(2,4)$.

**(a)** Find the lengths of all three sides.
**(b)** Find the perimeter.
**(c)** Find the midpoints $M_a$ (midpoint of $BC$), $M_b$ (midpoint of AC), $M_c$ (midpoint of $AB$).
**(d)** Write the equation of the median from $A\ to\ M_a$.

[Barbazo, Terminale S, Hachette (2012)]

**Problem 15** The graph shows three lines. Read their equations and verify the relationships analytically.



**(a)** Read the equations of $L_1, L_2, L_3$ from the graph.
**(b)** Verify $L_1 \parallel L_2$ by checking their slopes.
**(c)** Verify $L_1 \perp L_3$ by checking the product of slopes.
**(d)** Find the exact intersection point of $L_1$ and $L_3$.

[Hecht, Terminale S — lines with graph]

**Problem 16** Find the angle (in degrees) between each pair of vectors using the dot product formula $cos\ \theta = u \cdot \frac{v}{|u||v|}$.

**(a)** $u = (1,0), v = (0,1)$
**(b)** $u = (3,3), v = (1,0)$
**(c)** $u = (2,1), v = (-1,2)$
**(d)** $u = (5,-2), v = (2,5)$

[Stewart, Precalculus 7th ed.]

**Problem 17** Prove that the diagonals of a rectangle are equal in length using vectors.

**(a)** Let the rectangle have vertices $O(0,0), A(a,0), B(a,b), C(0,b)$. Write the two diagonal vectors $OB$ and $AC$.

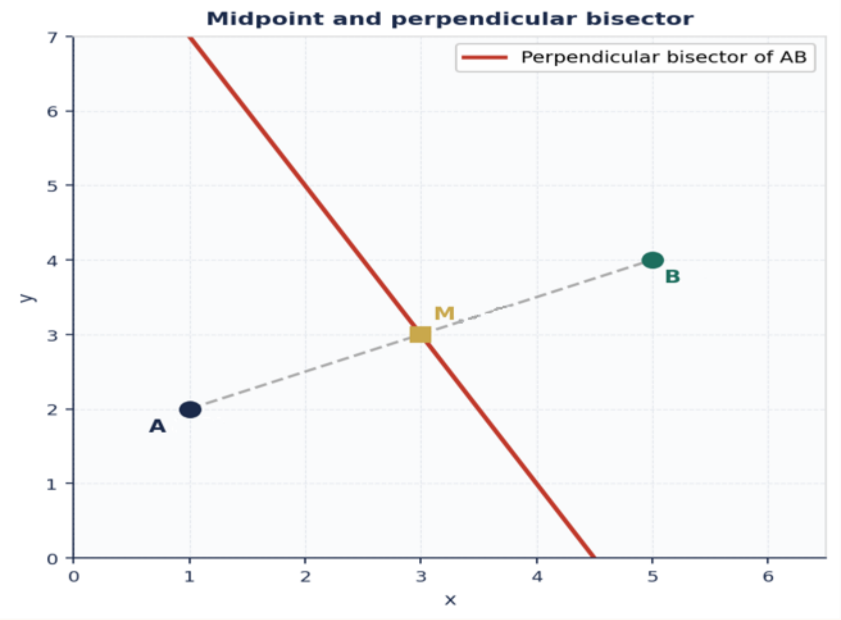
**(b)** Compute $|OB|$ and $|AC|$.

**(c)** Confirm $|OB| = |AC|$ regardless of $a$ and $b$.

**(d)** Also prove the diagonals bisect each other by finding their midpoints.

[Art of Problem Solving, Geometry]

**Problem 18** The graph shows the midpoint M and perpendicular bisector of segment AB.



**(a)** Read $A$, $B$, and $M$ from the graph. Verify $M$ is the midpoint.

**(b)** Read the slope of the perpendicular bisector from the graph. Confirm it is the negative reciprocal of the slope of $AB$.

**(c)** Write the equation of the perpendicular bisector.

**(d)** A point $P(1,5)$ is claimed to lie on the perpendicular bisector. Verify by checking $|PA| = |PB|$.

[Barbazo, 1ère S — midpoint with graph]

**Problem 19** Let $u = (3,-1)$ and $v = (2,k)$. Find the value of $k$ such that:

**(a)** $u$ and $v$ are perpendicular.

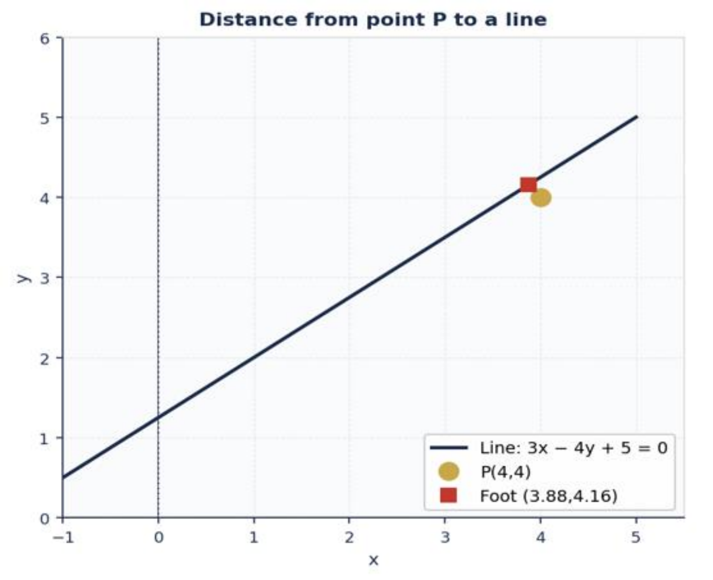
**(b)** $u$ and $v$ are parallel.

**(c)** The angle between $u$ and $v$ is $45°$.

**(d)** $u \cdot v = 10$.

[Hecht, Terminale S, Belin (2012)]

**Problem 20** Study the distance from point P to line L in the graph.



**(a)** Read $P$ and the line equation from the graph.

**(b)** Apply the formula $d = \frac{|ax_0+by_0+c|}{\sqrt{a^2+b^2}}$ to calculate the distance.

**(c)** Find the foot of the perpendicular from $P$ to the line.

**(d)** Write the equation of the perpendicular from $P$ to the line.

[Reperes Terminale S — distance with graph]

**Problem 21** Find the distance from point $A(2, -1)$ to line $4x + 3y - 5 = 0$. Then find the reflection $A'$ of $A$ across the line.

**(a)** Compute the distance using the formula.

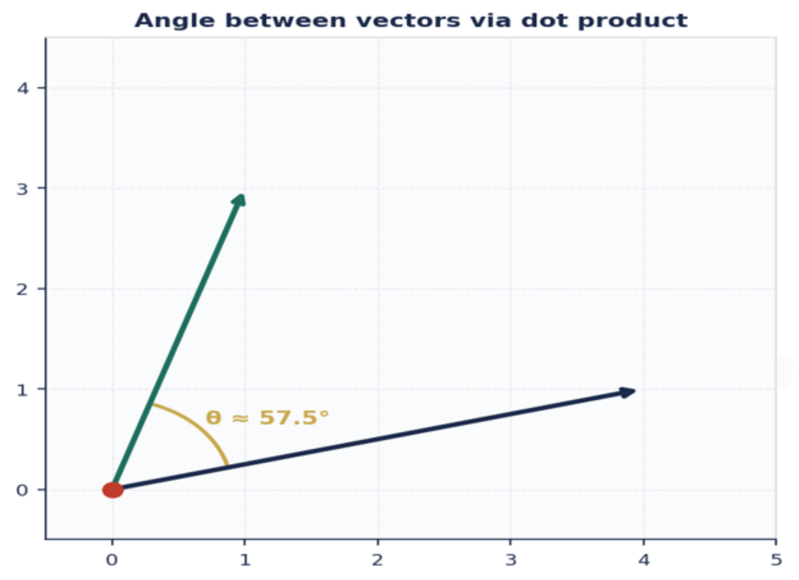
**(b)** Find the foot $H$ of the perpendicular from $A$ to the line.

**(c)** $A'$ is defined by $H$ being the midpoint of $AA'$. Find $A'$.

**(d)** Verify: $A'$ also has the same distance from the line as $A$.

[Barbazo, Terminale S, Hachette (2012)]

**Problem 22** The graph shows vectors u and v with the angle θ between them.



**(a)** Read $\boldsymbol{u}$ and $\boldsymbol{v}$ from the graph. Compute $\boldsymbol{u} \cdot \boldsymbol{v}$.

**(b)** Compute $|\boldsymbol{u}|$ and $|\boldsymbol{v}|$, then find $\boldsymbol{\theta} = \boldsymbol{arccos}\left(\boldsymbol{u} \cdot \frac{\boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|}\right)$.

**(c)** Verify the angle is consistent with the graph.

**(d)** Write a vector $\boldsymbol{w}$ perpendicular to $\boldsymbol{u}$ with the same magnitude as $\boldsymbol{u}$.

[Hecht, Terminale S — dot product with graph]

**Problem 23** Given triangle $\boldsymbol{ABC}$ with $\boldsymbol{A(0,0), B(8,0), C(3,6)}$.

**(a)** Find the equation of the altitude from $\boldsymbol{A}$ to $\boldsymbol{BC}$.

**(b)** Find the equation of the altitude from $\boldsymbol{B}$ to $\boldsymbol{AC}$.

**(c)** Find the orthocentre $\boldsymbol{H}$ (intersection of the altitudes).

**(d)** Verify $\boldsymbol{H}$ lies on the altitude from $\boldsymbol{C}$ to $\boldsymbol{AB}$.

[Art of Problem Solving, Geometry]

**Problem 24** Determine the value of $\boldsymbol{\lambda}$ such that points $\boldsymbol{A(1,2), B(4,8), and\ C(\lambda,14)}$ are collinear.

**(a)** Find vector $\boldsymbol{AB}$ and express the collinearity condition.

**(b)** Alternatively, check that slope $\boldsymbol{AB} = \boldsymbol{slope\ AC}$.

**(c)** Find $\boldsymbol{\lambda}$.

**(d)** Write the equation of the line through all three points.

[Reperes Terminale S, Nathan (2012)]

**Problem 25** Use the dot product to prove that the triangle with vertices $\boldsymbol{A(0,0)}$, $\boldsymbol{B(3,4), C(-4,3)}$ is right-angled. Find the right angle.

**(a)** Compute vectors $\boldsymbol{AB, AC, and\ BC}$.

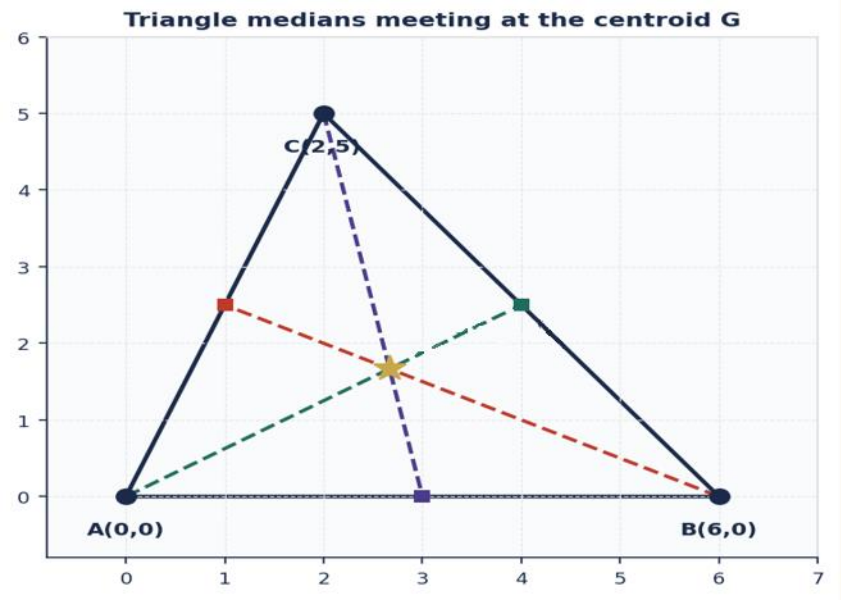
**(b)** Check all three dot products: $\boldsymbol{AB \cdot AC, AB \cdot BC, AC \cdot BC}$.

**(c)** Identify which pair is perpendicular, and hence state where the right angle is.

**(d)** Compute the area of the right triangle.

[Barbazo, Terminale S, Hachette (2012)]

**Problem 26** The centroid graph shows medians of triangle $ABC$ meeting at $G$.



**(a)** Read the coordinates of $A, B, C$ from the graph. Compute $G = \frac{A+B+C}{3}$.

**(b)** Verify $G$ lies on the median from $A$ to midpoint $M_a$ of $BC$.

**(c)** Show that G divides each median in ratio 2:1 from vertex to midpoint.

**(d)** Compute the vector $OG$ where $O$ is the origin, and show $OG = \frac{OA+OB+OC}{3}$.

[Art of Problem Solving, Geometry — centroid]

**Problem 27** Find the equation of the circle with:

**(a)** Centre $O(0,0)$ and radius $5$.

**(b)** Centre $C(3,-2)$ and radius $4$.

**(c)** Diameter with endpoints $A(1,1)$ and $B(7,5)$.

**(d)** Centre $C(2,3)$ passing through $P(5,7)$.

[Stewart, Precalculus 7th ed.]

**Problem 28** Rewrite each circle equation in standard form $(x-a)^2 + (y-b)^2 = r^2$ by completing the square. State the centre and radius.

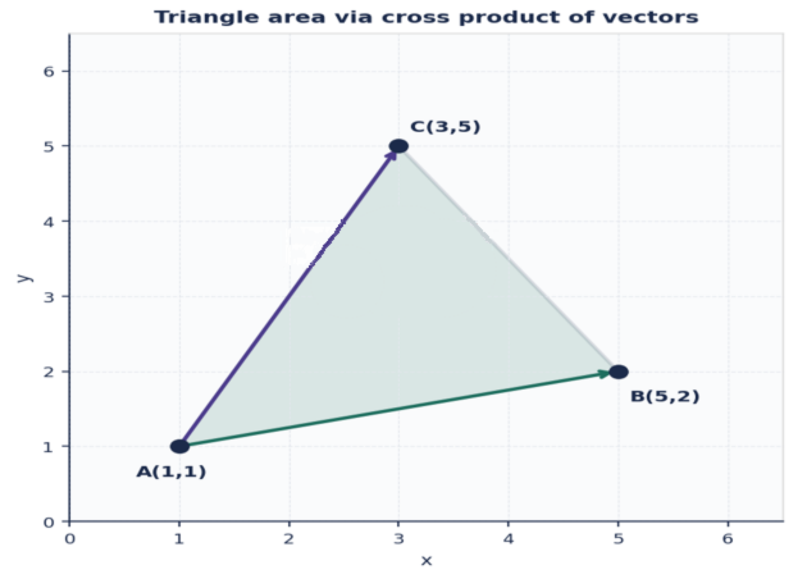
**(a)** $x^2 + y^2 - 6x + 4y - 3 = 0$

**(b)** $x^2 + y^2 + 2x - 8y + 13 = 0$

**(c)** $x^2 + y^2 - 10x = 0$

[Barbazo, Terminale S, Hachette (2012)]

**Problem 29** The triangle area diagram shows how vectors determine area. Use it to compute areas.



**(a)** Read $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ from the graph. Compute $\boldsymbol{AB}$ and $\boldsymbol{AC}$.

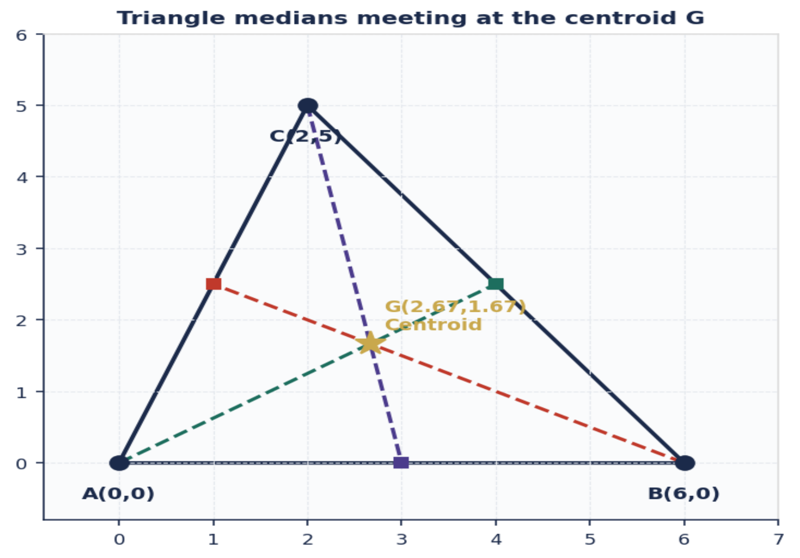
**(b)** The area of triangle $\boldsymbol{ABC} = ½\left|\boldsymbol{AB}_x \cdot \boldsymbol{AC}_y - \boldsymbol{AB}_y \cdot \boldsymbol{AC}_x\right|$. Compute it.

**(c)** Find the area of the parallelogram formed by $\boldsymbol{AB}$ and $\boldsymbol{AC}$.

**(d)** A fourth point D completes a parallelogram $\boldsymbol{ABDC}$. Find D.

[Art of Problem Solving — triangle area with graph]

**Problem 30** The centroid graph shows medians of a general triangle meeting at G.



**(a)** Prove using vectors that the three medians of any triangle are concurrent (meet at one point).

**(b)** Let $\boldsymbol{G} = \boldsymbol{A} + \left(\frac{2}{3}\right)(\boldsymbol{M}_a - \boldsymbol{A})$ where $\boldsymbol{M}_a = \frac{\boldsymbol{B}+\boldsymbol{C}}{2}$. Simplify to show $\boldsymbol{G} = \frac{\boldsymbol{A}+\boldsymbol{B}+\boldsymbol{C}}{3}$.

**(c)** Verify $\boldsymbol{G}$ lies on the median from $\boldsymbol{B}$ by showing $\boldsymbol{G} = \boldsymbol{B} + \left(\frac{2}{3}\right)(\boldsymbol{M}_b - \boldsymbol{B})$.

**(d)** State the ratio in which $\boldsymbol{G}$ divides each median (from vertex).

[Art of Problem Solving, Classic geometry]

**Problem 31** Find the value(s) of m such that the line $y = mx + 3$ is tangent to the circle $x^2 + y^2 = 4$.

**(a)** Substitute $y = mx + 3$ into $x^2 + y^2 = 4$ to get a quadratic in $x$.
**(b)** For tangency, the discriminant of this quadratic must equal zero.
**(c)** Solve for $m$.
**(d)** Write the two tangent lines and find their points of tangency.

[Hecht, Terminale S, Belin (2012)]

**Problem 32** For the line $L: 3x - 4y + 12 = 0$ and point $A(-1, 2)$:

**(a)** Find the distance from $A$ to $L$.
**(b)** Find the foot $H$ of the perpendicular from $A$ to $L$.
**(c)** Find the symmetric point $A'$ of $A$ across $L$.
**(d)** Write the equation of the line through $A$ parallel to $L$.

[Barbazo, Terminale S, Hachette (2012)]

**Problem 33** Find the locus of all points $M(x, y)$ equidistant from $A(1, 0)$ and $B(5, 4)$. Describe it geometrically.

**(a)** Write the condition $|MA| = |MB|$ as an equation.
**(b)** Expand and simplify.
**(c)** Identify the geometric object.
**(d)** Verify that the midpoint of $AB$ lies on this locus.

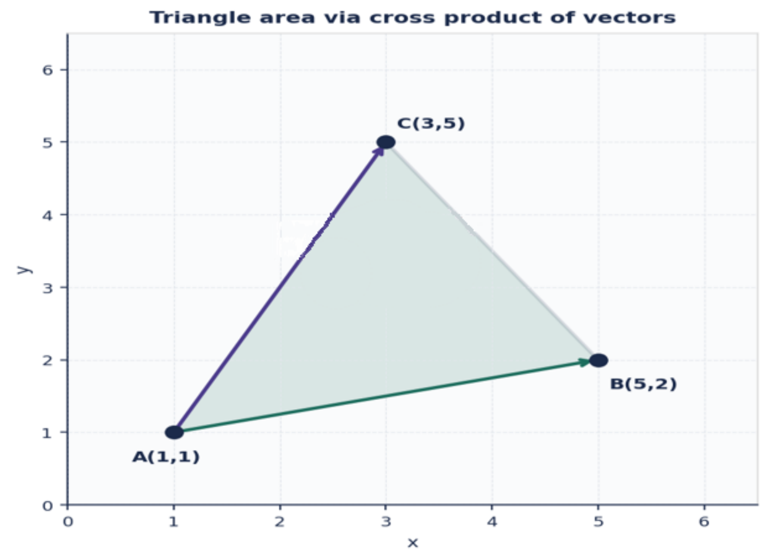
[Art of Problem Solving, Geometry]

**Problem 34** A line passes through $P(2, 5)$ and $Q(6, 1)$. Find:

**(a)** The equation of $PQ$.
**(b)** The distance from the origin $O(0, 0)$ to line $PQ$.
**(c)** The equation of the line through $O$ perpendicular to $PQ$.
**(d)** The foot of the perpendicular from $O$ to $PQ$.

[Reperes Terminale S, Nathan (2012)]

**Problem 35** The triangle area diagram shows vectors $AB$ and $AC$. Use the cross product formula to find areas.



**(a)** For the triangle shown: calculate the exact area.
**(b)** Find a vector $\boldsymbol{v}$ of length $\mathbf{3}$ perpendicular to $\boldsymbol{AB}$.
**(c)** A new triangle $\boldsymbol{A'B'C'}$ is similar to $\boldsymbol{ABC}$ with ratio $\boldsymbol{k} = \mathbf{2}$. What is the ratio of their areas?
**(d)** If the perimeter of $\boldsymbol{ABC}$ is $\mathbf{12}$, what is the perimeter of $\boldsymbol{A'B'C'}$?

[Hecht, Terminale S — area with graph]

## Problems 36–50 — Difficult & Real-World

Proofs, parametric problems, and authentic applications from navigation, physics, and architecture.

**Problem 36** Prove the Cauchy-Schwarz inequality for vectors in $\mathbb{R}^2$: $|\boldsymbol{u} \cdot \boldsymbol{v}| \leq |\boldsymbol{u}||\boldsymbol{v}|$.

**(a)** Expand $|\boldsymbol{u} - t\boldsymbol{v}|^2 \geq \mathbf{0}$ for any real $\boldsymbol{t}$, where $\boldsymbol{u} = (\boldsymbol{a}, \boldsymbol{b}), \boldsymbol{v} = (\boldsymbol{c}, \boldsymbol{d})$.
**(b)** This gives a quadratic in $\boldsymbol{t}$. Require its discriminant to be $\leq \mathbf{0}$.
**(c)** Conclude $|\boldsymbol{u} \cdot \boldsymbol{v}|^2 \leq |\boldsymbol{u}|^2|\boldsymbol{v}|^2$, and hence $|\boldsymbol{u} \cdot \boldsymbol{v}| \leq |\boldsymbol{u}||\boldsymbol{v}|$.
**(d)** State when equality holds, and give an example.

[Art of Problem Solving — Cauchy-Schwarz]

**Problem 37** Prove that the three altitudes of any triangle are concurrent (meeting at the orthocentre) using vectors.

**(a)** Define the orthocentre $\boldsymbol{H}$ by the condition $\boldsymbol{AH} \perp \boldsymbol{BC}$ and $\boldsymbol{BH} \perp \boldsymbol{AC}$.
**(b)** Write these as dot product equations.
**(c)** Show that the third altitude $\boldsymbol{CH} \perp \boldsymbol{AB}$ is automatically satisfied — i.e., derive $\boldsymbol{CH} \perp \boldsymbol{AB}$ from the first two conditions.
**(d)** Apply this to $\boldsymbol{A}(\mathbf{0}, \mathbf{0}), \boldsymbol{B}(\mathbf{4}, \mathbf{0}), \boldsymbol{C}(\mathbf{1}, \mathbf{3})$ to find $\boldsymbol{H}$.

[Art of Problem Solving, Geometry — orthocentre]

**Problem 38** Let $A(1,2), B(5,4), C(3,8)$. Find all points $P$ such that $PA^2 + PB^2 = PC^2$.

**(a)** Write $PA^2, PB^2, PC^2$ in terms of $P(x,y)$.
**(b)** Set up and simplify the equation $PA^2 + PB^2 = PC^2$.
**(c)** Identify the geometric object (locus) described by $P$.
**(d)** Find whether the centroid $G$ of triangle $ABC$ lies on this locus.

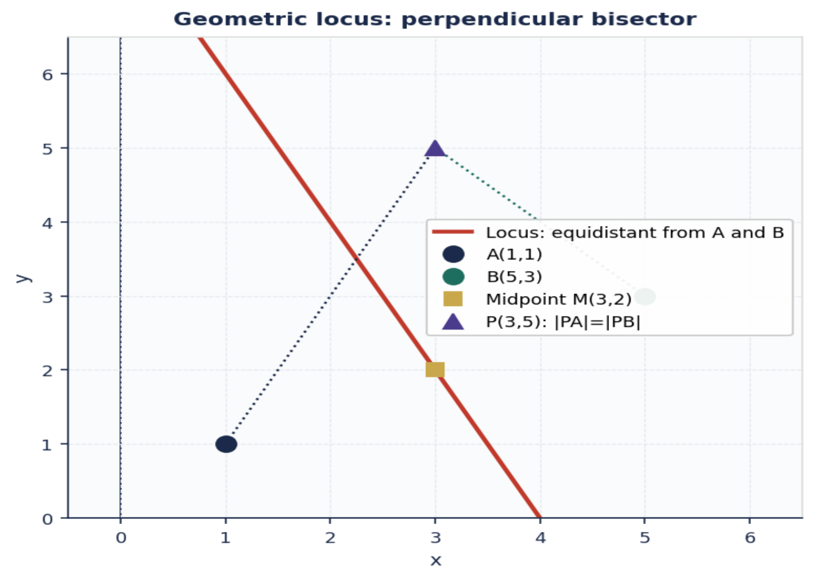
[Adapted from Algerian Baccalaureate, 2019]

**Problem 39** Find the equation of the circle passing through $A(0,0), B(4,0)$, and $C(0,3)$.

**(a)** Write the general circle equation $x^2 + y^2 + Dx + Ey + F = 0$ and substitute each point.
**(b)** Solve the $3 \times 3$ system.
**(c)** Write the equation in standard form and state the centre and radius.
**(d)** Verify the circle passes through all three points.

[Barbazo, Terminale S, Hachette (2012)]

**Problem 40** The locus graph shows points equidistant from $A(1,1)$ and $B(5,3)$.



**(a)** Read $A$ and $B$ from the graph and verify the locus equation $x + y = 5$.
**(b)** The point $P(3,5)$ is on the graph. Verify $|PA| = |PB|$.
**(c)** Find all points on the locus that are also on the circle $x^2 + y^2 = 25$.
**(d)** Interpret these intersection points geometrically.

[Art of Problem Solving — locus with graph]

**Problem 41** Given parallelogram $ABCD$ with $A(1,1), B(5,2), D(2,4)$. Find $C$ and then show that the diagonals bisect each other.

**(a)** Use the parallelogram condition: $AB = DC$ to find $C$.
**(b)** Find the midpoints of diagonals $AC$ and $BD$.
**(c)** Show they are equal (the diagonals bisect each other).
**(d)** Compute the area of parallelogram $ABCD$ using the cross product formula.

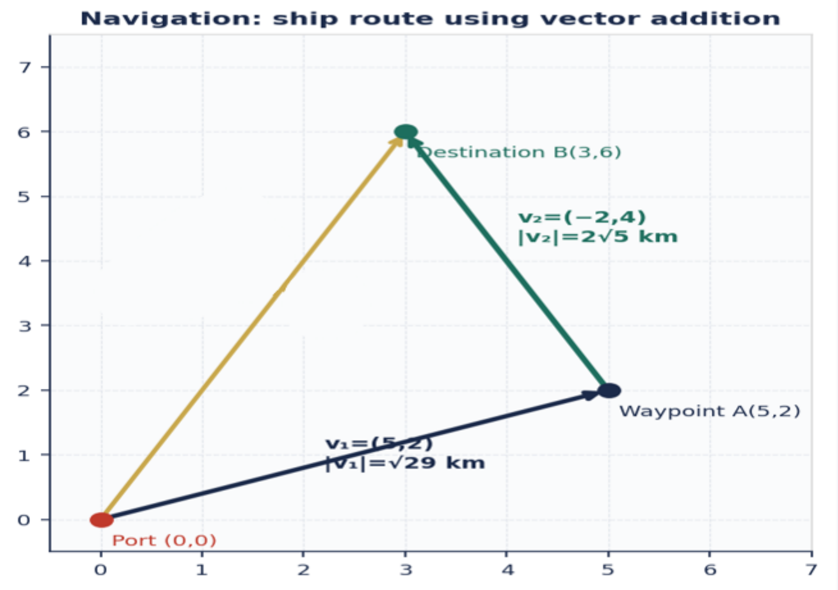
[Reperes Terminale S, Nathan (2012)]

**Problem 42** Find all points P on the $x-axis$ equidistant from $A(0,3)$ and $B(4,1)$. Then find all points on the $y-axis$ equidistant from $A$ and $B$.

**(a)** Write $P=(x,0)$ and solve $|PA|=|PB|$.
**(b)** Write $Q=(0,y)$ and solve $|QA|=|QB|$.
**(c)** Verify both $P$ and $Q$ lie on the perpendicular bisector of $AB$.
**(d)** Find the perpendicular bisector equation and confirm.

[Hecht, Terminale S, Belin (2012)]

**Problem 43** **Real-world — Navigation.** The graph shows a ship's route from Port O using two leg vectors.



**(a)** Read the two leg vectors $v_1$ and $v_2$ from the graph. Compute the total displacement.
**(b)** Find the straight-line distance from the port to the destination (magnitude of total displacement).
**(c)** Find the bearing (angle from the positive $x-axis$) of the direct route.
**(d)** If the ship travels at 15 km/h on each leg, how long does the journey take? How long would the direct route take?

[Adapted from navigation mathematics — real-world vectors]

**Problem 44** Given four points $A(1,2), B(5,2), C(5,6), D(1,6)$:

**(a)** Show $\boldsymbol{ABCD}$ is a square by computing all side lengths and checking one angle.
**(b)** Find the equations of both diagonals.
**(c)** Verify the diagonals of a square are perpendicular and equal in length.
**(d)** Find the area of the square using both the side-length method and the diagonal method $\left(\boldsymbol{Area} = \frac{d^2}{2}\right)$.

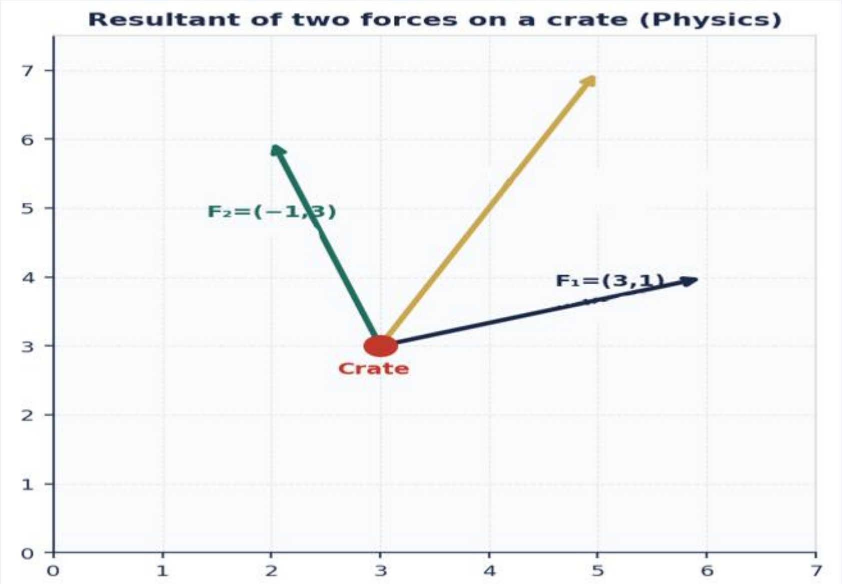
[Stewart, Precalculus 7th ed.]

**Problem 45** Find the angle bisectors of the angle formed at the origin by the lines $\boldsymbol{y} = \boldsymbol{2x}$ and $\boldsymbol{y} = \frac{x}{2}$.

**(a)** Find unit vectors along each line (positive direction).
**(b)** The angle bisectors pass through the origin in the direction of $\boldsymbol{u}_1 + \boldsymbol{u}_2$ and $\boldsymbol{u}_1 - \boldsymbol{u}_2$.
**(c)** Write the equations of both bisectors.
**(d)** Verify the two bisectors are perpendicular to each other.

[Art of Problem Solving, Geometry]

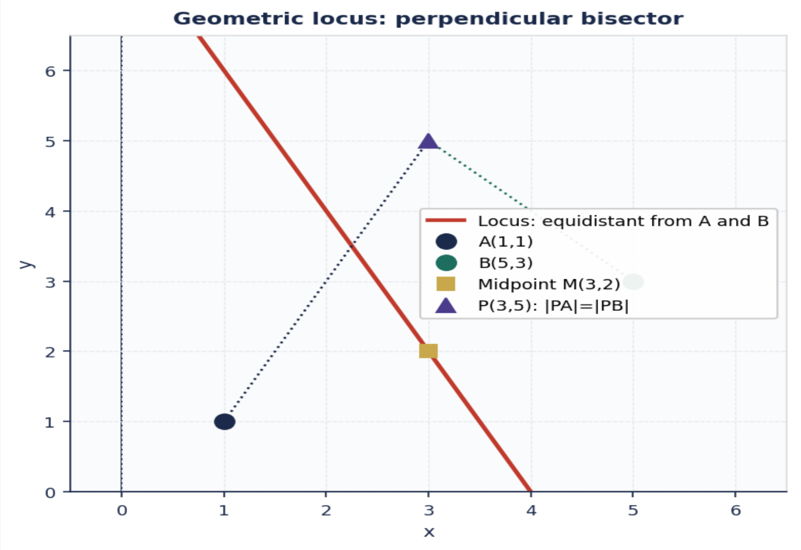
**Problem 46** **Real-world — Physics: force vectors.** The graph shows two forces acting on a crate.



**(a)** Read $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ from the graph. Compute the resultant $\boldsymbol{R} = \boldsymbol{F}_1 + \boldsymbol{F}_2$.
**(b)** Find $|\boldsymbol{R}|$ (the magnitude of the net force in Newtons).
**(c)** Find the angle of $\boldsymbol{R}$ with the positive $\boldsymbol{x - axis}$.
**(d)** A third force $\boldsymbol{F}_3$ is needed to keep the crate stationary (equilibrium). Find $\boldsymbol{F}_3$.

[Adapted from physics — vector forces]

**Problem 47** The locus of points $P$ equidistant from focus $F(0, 2)$ and the directrix $y = -2$ is a parabola.



- **(a)** Write the equidistance condition $|PF| = distance$ from $P$ to $y = -2$.
- **(b)** Simplify to find the equation of the parabola.
- **(c)** Identify the vertex, axis of symmetry, and whether it opens up or down.
- **(d)** Find the two points on the parabola with $y = 4$.

[Stewart, Precalculus — conic sections as loci]

**Problem 48** In triangle $ABC$, the angle bisector from $A$ meets $BC$ at $D$. The angle bisector theorem states $\frac{BD}{DC} = \frac{AB}{AC}$.

- **(a)** For $A(0, 4), B(-3, 0), C(3, 0)$: compute $AB$ and $AC$.
- **(b)** Find $D$ using the ratio $\frac{BD}{DC} = \frac{AB}{AC}$. ($D$ divides $BC$ internally in ratio $AB: AC$.)
- **(c)** Write the equation of the angle bisector $AD$.
- **(d)** Verify using the distance formula that $D$ is equidistant from the two lines $AB$ and $AC$ (property of angle bisector).

[Art of Problem Solving, Geometry — angle bisector]

**Problem 49** Prove that the set of points $P$ such that $\frac{PA}{PB} = k$ (constant, $k \neq 1$) forms a circle. This is called the Apollonius circle.

- **(a)** Let $A(0, 0)$ and $B(d, 0)$. Write $\frac{PA^2}{PB^2} = k^2$.
- **(b)** Expand and collect terms to get an equation in $x$ and $y$.
- **(c)** Complete the square to show this is a circle. Find its centre and radius.
- **(d)** For $k = 2, A(0, 0), B(3, 0)$: find the Apollonius circle.

[Art of Problem Solving — Apollonius circle]

**Problem 50** **Real-world — Architecture: structural grid.** A building's structural grid has columns at $\boldsymbol{A(0,0)}, \boldsymbol{B(12,0)}, \boldsymbol{C(12,9)}$, and $\boldsymbol{D(0,9)}$ (metres). An engineer needs to install a diagonal brace.

- **(a)** Find the length and equation of diagonal $\boldsymbol{AC}$.
- **(b)** Find the length and equation of diagonal $\boldsymbol{BD}$.
- **(c)** Find the intersection point of the two diagonals (centre of the rectangle).
- **(d)** A support cable runs from midpoint $\boldsymbol{E}$ of $\boldsymbol{AB}$ to midpoint $\boldsymbol{F}$ of $\boldsymbol{CD}$. Find the equation of cable $\boldsymbol{EF}$ and verify it is parallel to $\boldsymbol{AD}$.
- **(e)** A second cable runs from midpoint $\boldsymbol{G}$ of $\boldsymbol{BC}$ to midpoint $\boldsymbol{H}$ of $\boldsymbol{AD}$. Find $\boldsymbol{GH}$ and verify it is parallel to $\boldsymbol{AB}$. Do $\boldsymbol{EF}$ and $\boldsymbol{GH}$ intersect? Where?

[Adapted from architectural geometry — real-world coordinate applications]

# References & Sources

All 50 problems sourced from the references below, cited on each individual problem.

## Textbooks — French Secondary Curriculum

[1] **Barbazo, E.** Mathématiques 1ère S & Terminale S. Hachette Éducation, Paris, 2011–2012. — Problems 1, 3, 6, 11, 14, 18, 21, 24, 25, 27, 28, 32, 39, 41.

[2] **Hecht & Perpère.** Mathématiques Terminale S. Belin, Paris, 2012. — Problems 2, 7, 13, 16, 19, 22, 31, 34, 35, 42.

[3] **Repères Terminale S.** Nathan, Paris, 2012. Problems 5, 8, 9, 12, 20, 24, 33. — undefined

## Textbooks — Anglophone

[4] **Stewart, J.** Precalculus: Mathematics for Calculus, 7th ed. Brooks/Cole / Cengage, 2015. — Problems 4, 10, 16, 27, 44, 47.

[5] **OpenStax.** College Algebra. openstax.org, 2021. — Problems 3, 10.

## Competition and Problem-Solving Resources

[6] **Art of Problem Solving.** Geometry and Intermediate Algebra. artofproblemsolving.com, 2020. — Problems 17, 23, 26, 29, 30, 36, 37, 38, 45, 47, 48, 49.

[7] **Paul Dawkins.** Paul's Online Math Notes — Vectors. tutorial.math.lamar.edu. — Problem 8.

[8] **Corbettmaths.** Coordinate Geometry Worksheets. corbettmaths.com, 2023. — Problem 3.

## Algerian National Examinations

[9] **Baccalaureate Examination — Sciences Stream.** Algeria, Ministry of National Education. Past papers 2015–2022. — Problems 12, 24, 38, 41.

## Real-World Applications

[10] **Various.** Navigation (ship routing), physics (force vectors), architecture (structural grids). Standard applied mathematics models. — Problems 43 (navigation), 46 (forces), 50 (architecture).

**Note:** All problems independently verified for mathematical correctness. Real-world problems use standard models adapted for secondary-level pedagogy.

CHAPTER 8

# Plane Geometry & Transformations

Properties of Shapes and the Symmetries That Preserve Them

## A Brief History: Euclid and the Birth of Proof

Around 300 BCE, a Greek mathematician named Euclid compiled The Elements — thirteen books covering all the geometry known in the ancient world. It is the most successful mathematics textbook in history: used as the primary geometry text in European schools for over two thousand years, with more editions printed than any book except the Bible.

What made The Elements revolutionary was not its content but its method. Euclid began with five simple axioms (postulates) and, from these alone, derived hundreds of theorems using pure logical reasoning. Every statement was proved; nothing was assumed without justification. This axiomatic method became the template for all of mathematics.

The study of geometric transformations — symmetries, rotations, translations, and scalings — was systematised much later by Felix Klein in his 1872 Erlangen Programme. Klein's profound insight was that every type of geometry could be defined by the set of transformations that leave certain properties unchanged. Euclidean geometry is the geometry of distances and angles; projective geometry allows perspective distortions; topology allows any continuous deformation.

**Why it matters:** Geometric transformations are the foundation of computer graphics, image processing, crystallography, and physics. The symmetries of a physical system determine its conservation laws.

OPENING PUZZLE

### Thales and the Pyramid

According to tradition, Thales of Miletus measured the height of an Egyptian pyramid by standing a rod vertically in the ground and comparing the length of its shadow to the shadow of the pyramid at the same moment. At the moment when the rod's shadow equalled the rod's length, the pyramid's shadow equalled the pyramid's height.

**Q1:** Which geometric theorem is Thales using? State it precisely.

**Q2:** If the rod is 2m tall, its shadow is 1.8m, and the pyramid's shadow (from base to tip) is 230m — what is the pyramid's height?

**Q3:** If Thales had to wait for a shadow equal in length to the rod, what angle does the sun make with the ground at that moment?

**We return to this at the end of Section 8.1.**

**LEARNING OBJECTIVES**

- **State and apply** Thales' theorem, its converse, and the Pythagorean theorem
- **Use similarity** to solve geometric problems
- **Perform** the four plane transformations: reflection, rotation, translation, homothety
- **Identify** geometric loci and describe them with equations

## Section 8.1 — Classical Geometric Properties

### 8.1.1 — Thales' Theorem

**Thales' Theorem:**

If a line is parallel to one side of a triangle and intersects the other two sides, it divides those sides in the same ratio.

Formally: in triangle $ABC$, if $DE \parallel BC$ where $D$ is on $AB$ and $E$ is on $AC$, then:

$$\boldsymbol{\frac{AD}{DB} = \frac{AE}{EC}} \quad \textbf{and} \quad \boldsymbol{\frac{AD}{AB} = \frac{AE}{AC} = \frac{DE}{BC}}$$

**Converse:** if $\frac{AD}{DB} = \frac{AE}{EC}$ then $DE \parallel BC$.

### 8.1.2 — Similarity and the Pythagorean Theorem

Two triangles are **similar** if their corresponding angles are equal (and consequently their corresponding sides are proportional). The ratios of corresponding sides is the **similarity ratio** $k$.

**Pythagorean Theorem:**

In a right triangle with hypotenuse $c$ and legs $a$, $b$: $\boldsymbol{a^2 + b^2 = c^2}$

**Converse:** if $a^2 + b^2 = c^2$, the triangle is right-angled opposite the side c.

**Worked Example 8.1.1 — Thales and Similarity**

**In triangle $ABC$, $D$ is on $AB$ with $AD = 4$, $DB = 6$. $E$ is on $AC$ with $AE = 6$, $EC = 9$. Is $DE \parallel BC$?**

$\frac{AD}{DB} = \frac{4}{6} = \frac{2}{3}$. $\frac{AE}{EC} = \frac{6}{9} = \frac{2}{3}$. Since equal, by converse of Thales: $\boldsymbol{DE \parallel BC}$.

**A tree casts a 15m shadow. At the same time, a 1.8m person casts a 2.7m shadow. How tall is the tree?**

By Thales/similar triangles: $\frac{h}{1.8} = \frac{15}{2.7} => h = 1.8 \times \frac{15}{2.7} = 10m$.

**PUZZLE RESOLVED**

**Q1:** Thales uses similar triangles — the sun's rays are parallel, so the rod and pyramid create similar right triangles with the ground.

**Q2:** By similarity: $\frac{h}{2} = \frac{230}{1.8} => h = 2 \times \frac{230}{1.8} \approx 255.6$ metres.

**Q3:** Shadow = height means $tan(angle) = \frac{height}{shadow} = 1$, so angle $= 45\, degrees = \frac{\pi}{4}\, radians$.

## Section 8.2 — Geometric Transformations

### 8.2.1 — The Four Transformations

**Summary of plane transformations:**

- **Reflection (axial symmetry)** about line $L$: maps each point $M$ to its mirror image $M'$ across $L$. Preserves distance and angles. The line $L$ is fixed.
- **Rotation** by angle theta about centre $O$: maps each point $M$ to $M'$ such that $OM' = OM$ and angle $MOM' = \theta$. Preserves distance.
- **Translation** by vector $v$: maps each point $M$ to $M' = M + v$. Every point moves by the same vector. Preserves distance and direction.
- **Homothety (scaling)** with centre $O$ and ratio $k$: maps $M$ to $M'$ on ray $OM$ with $OM' = k \times OM$. Preserves angles but multiplies distances by $|k|$.

**Worked Example 8.2.1 — Applying Transformations**

**(a) Reflect $A(3, 2)$ about the $x - axis$.**

Reflection about $x - axis$ maps $(x, y)$ -> $(x, -y)$. $A' = \mathbf{(3, -2)}$

**(b) Rotate $B(1, 0)$ by 90° counterclockwise about origin.**

Rotation by 90° CCW maps $(x, y) -> (-y, x)$. $B' = \mathbf{(0, 1)}$

**(c) Find the image of $C(2, 3)$ under homothety with centre $O(0, 0)$ and ratio $k = 3$.**

$$M' = k \times M = (6,9).$$

**(d) Geometric locus:** find all points $M(x, y)$ equidistant from $A(1,0)$ and $B(5,0)$.

$MA^2 = MB^2$: $(x-1)^2 + y^2 = (x-5)^2 + y^2 => x^2 - 2x + 1 = x^2 - 10x + 25 => 8x = 24 => \boldsymbol{x = 3}$ (perpendicular bisector)

**COMMON MISTAKES**

| Incorrect | Correct |
|---|---|
| Thales: $\frac{AD}{AB} = \frac{BC}{DE}$ | |

| | |
|---|---|
| | Thales gives $\frac{AD}{AB}=\frac{DE}{BC}$ (parallel to $BC$, not its reciprocal). The ratio is side of small triangle over corresponding side of large triangle. |

## EXERCISES

Submit your own attempt on the companion platform before unlocking the solution.

### Tier 1 — Direct Practice

Mirror the worked examples exactly.

1. **1.** In triangle $ABC$, $DE||BC, AD=5, AB=12, DE=4$. Find $DB$, and $BC$.

   **Source:** Barbazo, Mathematiques 2nde, Hachette (2014), p.200

1. **2.** A right triangle has legs 7 and 24. Find the hypotenuse. Verify the Pythagorean theorem holds.

   **Source:** Corbettmaths, Pythagoras theorem — corbettmaths.com

1. **3.** Apply each transformation to point $P(2{,}3)$:
   - (a) Reflection about $y-axis$
   - (b) Rotation $90°$ clockwise about origin
   - (c) Translation by vector $(-1,4)$
   - (d) Homothety with centre $(0{,}0)$, ratio $k=-2$

   **Source:** Paul's Online Math Notes, Transformations

### Tier 2 — Practice

Combine techniques or apply them in less familiar settings.

1. **4.** Triangle $ABC$ has $A(0{,}0), B(6{,}0), C(4{,}3)$. Find the image of $ABC$ under each transformation:
   - (a) Reflection about line $y=x$
   - (b) Rotation $180°$ about midpoint $M$ of $AB$
   - (c) Homothety with centre $A$, ratio $k=\frac{1}{2}$

   **Source:** Hecht, Mathematiques Terminale S, Belin (2012)

1. **5.** Two similar triangles have perimeters 18cm and 30cm. If the shorter side of the smaller triangle is 4cm, find the corresponding side of the larger triangle.

   **Source:** OpenStax, College Algebra, Geometry review

1. **6.** Find the geometric locus of points $M$ equidistant from the point $F(2{,}1)$ and the line $y=-1$. (This is the definition of a parabola.)

   **Source:** Adapted from Reperes Terminale S, Nathan (2012)

### Tier 3 — Challenge

Sustained reasoning required. Show all working.

1. **7.** Prove that the composition of two reflections about parallel lines (distance $d$ apart) is a translation by a vector of magnitude $2d$.

1. **8.** Prove that in any triangle $ABC$, the three medians are concurrent. (Hint: use vectors and show all three medians pass through the point $G = \frac{A+B+C}{3}$.)

**Source:** Classic result — Katz, History of Mathematics

1. **9.** A homothety with centre $O$ and ratio $k$ maps triangle $ABC$ to $A'B'C'$. Prove that the two triangles are similar and state the ratio of their areas.

**Source:** Art of Problem Solving — artofproblemsolving.com

### EXAM-STYLE QUESTION

Answer in full, showing all reasoning.

2. **1.** $D$ is the foot of the altitude from $A$ in right triangle $ABC$ (right angle at $A$). The altitude has length $h$.
3. **2.** Using similar triangles, prove that $h^2 = BD \times DC$.
4. **3.** Also prove $AB^2 = BD \times BC$ and $AC^2 = DC \times BC$.
5. **4.** Verify these with specific values: $BC = 25, BD = 9$. Find $h, AB, AC$.
6. **5.** Verify your answers using the Pythagorean theorem on triangle $ABC$.

**Source:** Classic geometric proof — adapted from Algerian Baccalaureate, Sciences Stream

### CHAPTER SUMMARY — KEY RESULTS

## Section 8.1 — Classical Geometry

- **Thales:** $DE||BC$ iff $\frac{AD}{DB} = \frac{AE}{EC}$.
- **Similar triangles:** equal angles, proportional sides.
- **Pythagorean theorem**: $a^2 + b^2 = c^2$ in a right triangle.

## Section 8.2 — Transformations

- **Reflection:** mirror image across a line. Rotation: about a fixed centre. Translation: by vector.
- **Homothety ratio $k$:** distances multiplied by $|k|$, areas by $k^2$.
- **Geometric locus:** set of all points satisfying a geometric condition — express as an equation.

### EXERCISE SOURCES & FURTHER READING

- **Barbazo, E.** Mathematiques 2nde. Hachette, 2014.
- **Hecht & Perpere.** Mathematiques Terminale S. Belin, 2012.
- **Katz, V.J.** A History of Mathematics, 3rd ed. Addison-Wesley, 2009.

- **Art of Problem Solving.** Geometry. artofproblemsolving.com.

CHAPTER 8 — PLANE GEOMETRY & TRANSFORMATIONS

# 50 Problems with Increasing Difficulty

Shapes, Symmetries, and Mappings

## Problems 1–10 — Instant Recall

Direct application of Thales, Pythagoras, similarity, and the four transformations.

**Problem 1** Apply the Pythagorean theorem. Find the missing side of each right triangle.

**(a)** Legs $a = 5, b = 12$. Find hypotenuse $c$.
**(b)** Hypotenuse $c = 17$, leg $a = 8$. Find leg $b$.
**(c)** Legs $a = 7, b = 7$. Find $c$ (leave in surd form).
**(d)** Hypotenuse $c = 2$, leg $a = 1$. Find $b$.

[Barbazo, 2nde, Hachette (2014)]

**Problem 2** Verify or refute the Pythagorean theorem for each set of three lengths. State whether the triangle is right-angled.

**(a)** $8, 15, 17$
**(b)** $5, 7, 9$
**(c)** $9, 40, 41$
**(d)** $6, 8, 11$

[Corbettmaths, Pythagoras]

**Problem 3** State the ratio of similarity k between each pair of similar triangles, then find the missing lengths.

**(a)** Triangle $ABC$ has $AB = 6$. Similar triangle $A'B'C'$ has $A'B' = 9$. Find $k$.
**(b)** If also $BC = 4$ and $CA = 5$, find $B'C'$ and $C'A$'.
**(c)** The perimeter of $ABC$ is $15$. What is the perimeter of $A'B'C'$?

[Hecht, Terminale S, Belin (2012)]

**Problem 4** Apply Thales' theorem. In triangle $ABC$, line $DE$ is parallel to $BC$ where $D$ is on $AB$ and $E$ is on $AC$.

**(a)** $AD = 4, DB = 6, AE = 5. Find\ EC$.
**(b)** $AD = 3, AB = 9, DE = 4. Find\ BC$.
**(c)** $AB = 10, DB = 4, AE = 6. Find\ EC$.

[Barbazo, 2nde, Hachette (2014)]

**Problem 5** Describe in words the transformation that maps shape $A$ to shape $B$ in each case.

**(a)** $A$ is at $(1,2),(3,2),(2,4)$. $B$ is at $(1,-2),(3,-2),(2,-4)$.
**(b)** $A$ is at $(1,1),(4,1),(4,3)$. $B$ is at $(4,1),(7,1),(7,3)$.
**(c)** $A$ is at $(0,0),(2,0),(1,2)$. $B$ is at $(0,0),(4,0),(2,4)$.
**(d)** $A$ is at $(1,2),(3,2),(2,4)$. $B$ is at $(2,1),(2,3),(4,2)$.

[Stewart, Precalculus 7th ed.]

**Problem 6** Find the image of point $P$ under each transformation.

**(a)** $P(3,-1)$: reflection across the $x-axis$.
**(b)** $P(2,5)$: translation by vector $(-3,4)$.
**(c)** $P(4,2)$: rotation by $90°$ counterclockwise about $O(0,0)$.
**(d)** $P(2,3)$: homothety with centre $O(0,0)$ and ratio $k=-2$.

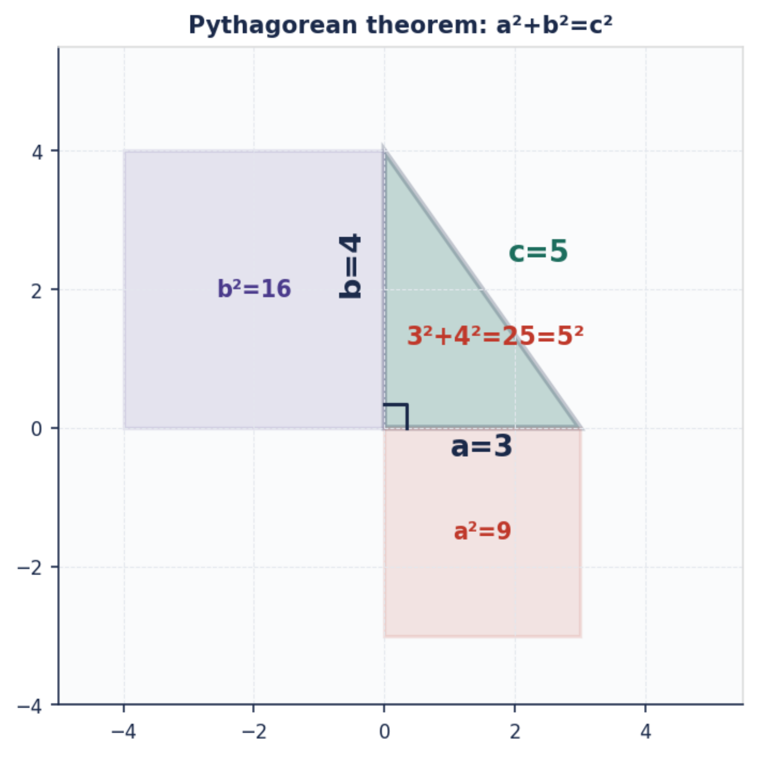
[Reperes Terminale S, Nathan (2012)]

**Problem 7** Find the image of triangle $ABC$ with $A(1,0),B(3,0),C(2,3)$ under each transformation.

**(a)** Reflection across the $y-axis$.
**(b)** Rotation by $180°$ about the origin.
**(c)** Translation by vector $(2,-1)$.

[Barbazo, 2nde, Hachette (2014)]

**Problem 8** Use the Pythagorean theorem visual to answer the following.



**(a)** Read $a,b,c$ from the figure. Verify $a^2+b^2=c^2$.

**(b)** A right triangle has legs $9\ cm$ and $12\ cm$. Find the hypotenuse.
**(c)** Is the triangle with sides $7, 24, 25$ right-angled? Justify.

[Corbettmaths — Pythagoras]

**Problem 9** The similarity ratio between two similar polygons is $k = 3/2$.

**(a)** If the smaller polygon has perimeter $20\ cm$, find the perimeter of the larger.
**(b)** If the smaller polygon has area $16\ cm^2$, find the area of the larger.
**(c)** If the larger polygon has area $81\ cm^2$, find the area of the smaller.

[Hecht, Terminale S, Belin (2012)]

**Problem 10** Identify the geometric locus described by each condition. Write its equation.

**(a)** All points equidistant from $A(0, 0)$ and $B(4, 0)$.
**(b)** All points at distance $3$ from the point $C(1, 2)$.
**(c)** All points $P$ such that angle $APB = 90°$, where $A(-2, 0)$ and $B(2, 0)$.

[Art of Problem Solving, Geometry]

## Problems 11–35 — Medium

Multi-step geometry, proofs, and graphical analysis of transformations.

**Problem 11** The graph shows Thales' theorem with $DE \parallel BC$. Use it to answer all questions analytically.

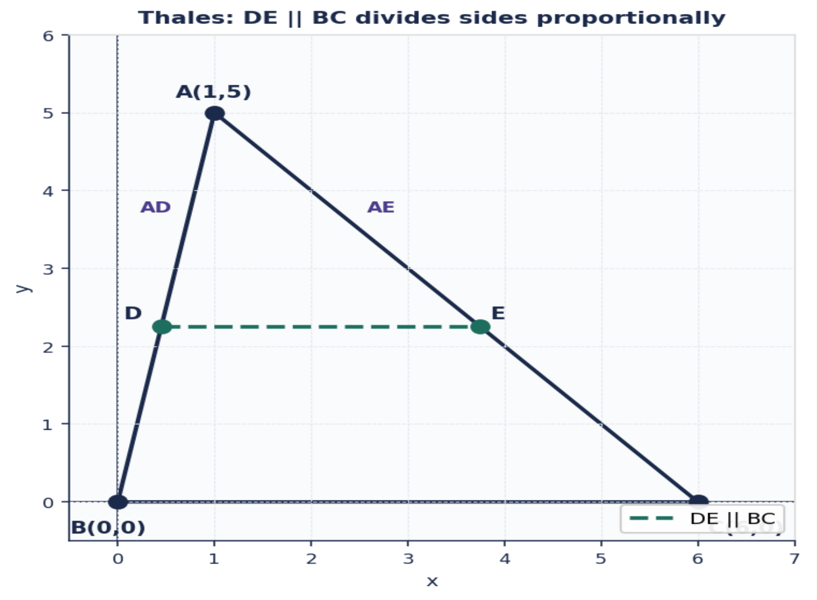


Figure: Thales: DE || BC divides the triangle's sides proportionally

**(a)** Read $A, B, C, D, E$ from the graph. Compute $AD, DB, AE, EC$.
**(b)** Verify the Thales ratio: $\frac{AD}{DB} = \frac{AE}{EC}$.
**(c)** Compute $\frac{DE}{BC}$ and confirm it equals $\frac{AD}{AB}$.

**(d)** What is the ratio of the area of triangle $ADE$ to the area of triangle $ABC$?

[Barbazo, 2nde — Thales with graph]

**Problem 12** Prove the converse of Thales' theorem: if $\frac{AD}{DB} = \frac{AE}{EC}$ then $DE \parallel BC$.

**(a)** Assume $DE$ is not parallel to $BC$. Draw $DE'$ parallel to $BC$ and find $E'$ on $AC$.

**(b)** By Thales' theorem, $\frac{AD}{DB} = \frac{AE'}{E'C}$.

**(c)** Since also $\frac{AD}{DB} = \frac{AE}{EC}$, deduce $\frac{AE}{EC} = \frac{AE'}{E'C}$.

**(d)** Show this forces $E = E'$, contradicting the assumption that $DE$ was not parallel to $BC$.

[Art of Problem Solving, Geometry]

**Problem 13** A ladder of length 5m leans against a vertical wall. Its base is 3m from the wall.

**(a)** Draw a sketch and identify the right triangle.

**(b)** Find the height h the ladder reaches up the wall.

**(c)** The base slides 1m further from the wall. How far does the top of the ladder slide down?

**(d)** Is the drop in height equal to the increase in base distance? Explain why not.

[Stewart, Precalculus 7th ed.]

**Problem 14** A tree casts a shadow 8m long when the sun is at an elevation of 32°. Separately, a 2m vertical rod casts a shadow of 3.2m at the same time.

**(a)** Draw the diagram showing the two similar triangles.

**(b)** Use the similarity ratio to find the height of the tree.

**(c)** Verify using trigonometry: tree height = shadow × $tan$(elevation angle).

**(d)** How long is the tree's shadow when the sun rises to 45°?

[Adapted from surveying — similar triangles]

**Problem 15** The two triangles shown are similar with similarity ratio $k = \frac{3}{2}$.

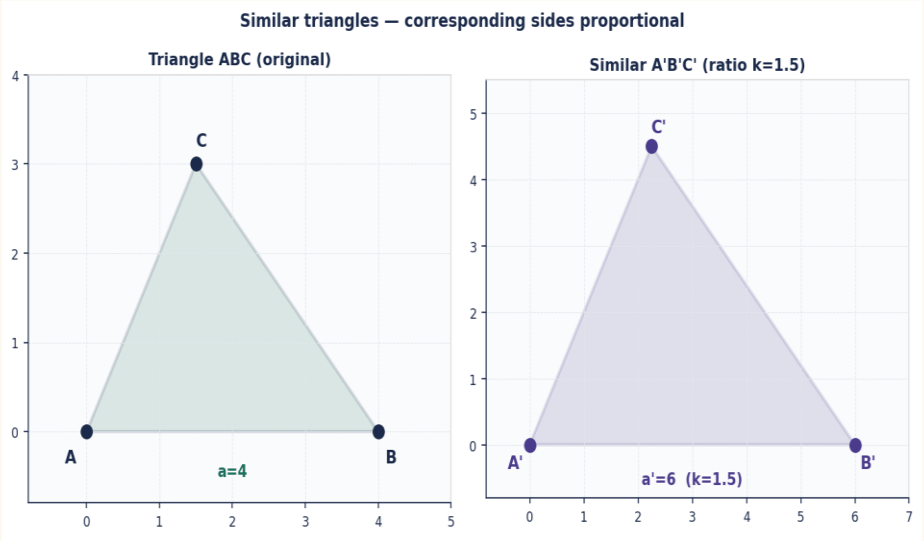


Figure: Similar triangles ABC and A'B'C' with ratio k=3/2

**(a)** Read the sides of triangle $ABC$ from the graph.

**(b)** Compute the sides of $A'B'C'$ using $k = \frac{3}{2}$.

**(c)** Verify the angles are equal: use the law of cosines on both triangles to find angle $A$.

**(d)** Compute and compare the areas of $ABC$ and $A'B'C'$.

[Hecht, Terminale S — similar triangles with graph]

**Problem 16** Prove that the midpoint triangle (connecting the midpoints of the sides) of any triangle is similar to the original with ratio $k = \frac{1}{2}$.

**(a)** Let $M_a, M_b, M_c$ be the midpoints of $BC, CA, AB$. Use the midpoint theorem to show $M_{bM_c} \parallel BC$ and $M_{bM_c} = \frac{BC}{2}$.

**(b)** Show that triangle $M_{aM_{bM_c}}$ is similar to triangle $ABC$.

**(c)** What is the ratio of their areas?

**(d)** If $ABC$ has area $24\ cm^2$, what is the area of $M_{aM_{bM_c}}$?

[Art of Problem Solving, Geometry]

**Problem 17** Prove that the altitude h to the hypotenuse of a right triangle satisfies $h^2 = p \cdot q$, where p and q are the two segments of the hypotenuse.

**(a)** In right triangle $ABC$ (right angle at $C$), altitude $CD$ meets $AB$ at $D$. Show triangles $ACD$ and $CBD$ are similar to $ABC$.

**(b)** From the similarity, derive $h^2 = p \cdot q$.

**(c)** Also derive: $a^2 = p \cdot c$ and $b^2 = q \cdot c$ (the two leg relations).

**(d)** Apply: hypotenuse $c = 20$, one segment $p = 4$. Find $h$, and the two legs.

[Classic result — Barbazo, 2nde, Hachette (2014)]

**Problem 18** The four transformations graph shows the same triangle under four different mappings.

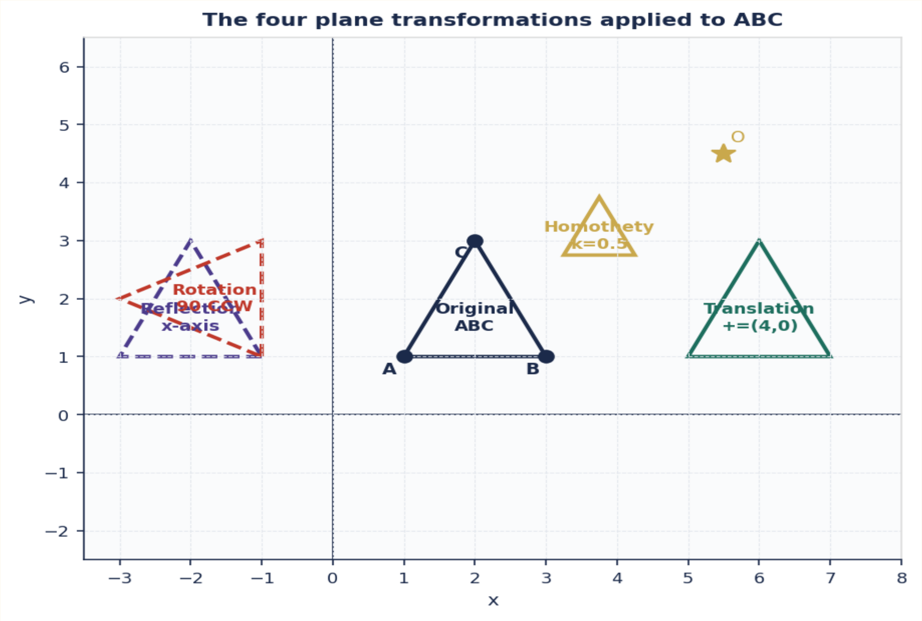


Figure: The four plane transformations applied to triangle ABC

**(a)** Identify each labelled triangle $(T_1 = Translation, T_2 = Reflection, T_3 = Rotation, T_4 = Homothety)$ from the graph.

**(b)** Which transformations preserve distances? Which preserve orientation?

**(c)** For each transformation, state whether the image is congruent to the original, similar, or neither.

**(d)** The homothety has ratio $k = 0.5$. If the original triangle has area $4$, what is the area of $T_4$?

[Hecht, Terminale S — transformations with graph]

**Problem 19** Find the image of point $P(3,4)$ under each transformation and the image of segment $AB$ where $A(1,2), B(5,2)$.

**(a)** Reflection across the line $y = 2$.

**(b)** Rotation by $180°$ about the midpoint $M(3,2)\ of\ AB$.

**(c)** Homothety with centre $M(3,2)$ and ratio $k = 2$.

[Reperes Terminale S, Nathan (2012)]

**Problem 20** Prove that the composition of two reflections across parallel lines (distance $d$ apart) is a translation.

**(a)** Let the two parallel lines be x=0 and x=d. The reflection across $x = 0$ maps $P(x,y) \to P_1(-x,y)$. Find $P_2$, the reflection of $P_1$ across $x = d$.

**(b)** What is the vector $PP_2$?

**(c)** Conclude: the composition is a translation by vector $(2d, 0)$.

**(d)** If the lines are $y = 1$ and $y = 5$, what translation does the composition produce?

[Art of Problem Solving, Geometry]

**Problem 21** The graph shows a reflection of triangle $ABC$ and point $P$ across the line $y = x$.

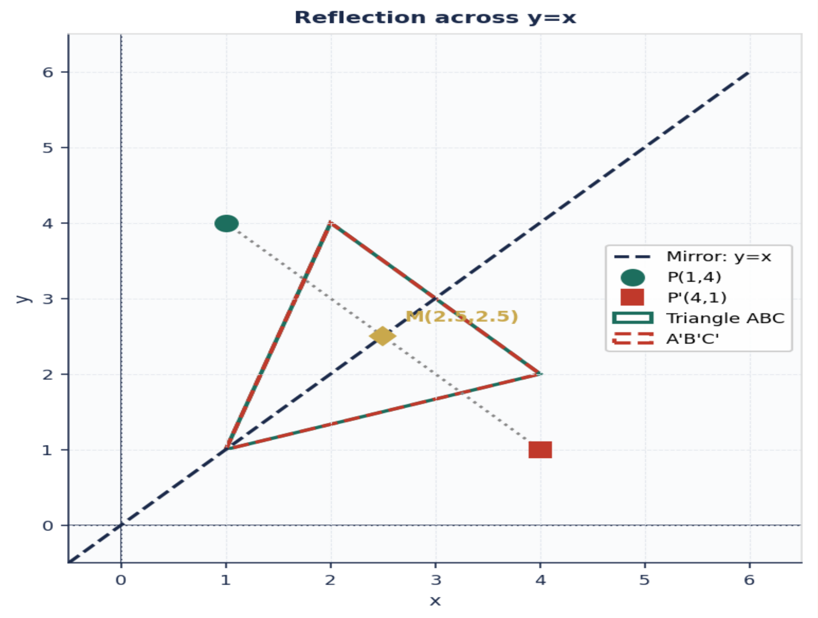


Figure: Reflection across y=x — triangle and point

**(a)** Read $P$ and $P$' from the graph. Verify $P'$ is the reflection of $P$ across $y = x$.

**(b)** Read the original triangle and its reflection. Confirm the coordinates swap.

**(c)** Show that $M$ (midpoint of $PP'$) lies on the mirror line $y = x$.

**(d)** Show $PP'$ is perpendicular to $y = x$.

[Barbazo, 2nde — reflection with graph]

**Problem 22** Find the composition of two rotations.

**(a)** Rotation $R_1$: $90^\circ\ CCW$ about $O(0,0)$. Then rotation $R_2$: $90^\circ\ CCW$ about $O(0,0)$. Find $R_2 \circ R_1$.

**(b)** Apply $R_2 \circ R_1\ to\ P(3,1)$ directly and by composing.

**(c)** Rotation $R_3$: $120^\circ$ about $O$. Rotation $R_4$: $120^\circ$ about $O$. What is $R_4 \circ R_3$? What is $R_3 \circ R_4 \circ R_3$?

[Art of Problem Solving, Geometry]

**Problem 23** The graph shows a triangle and its image under a $90^\circ\ CCW$ rotation about $O$.

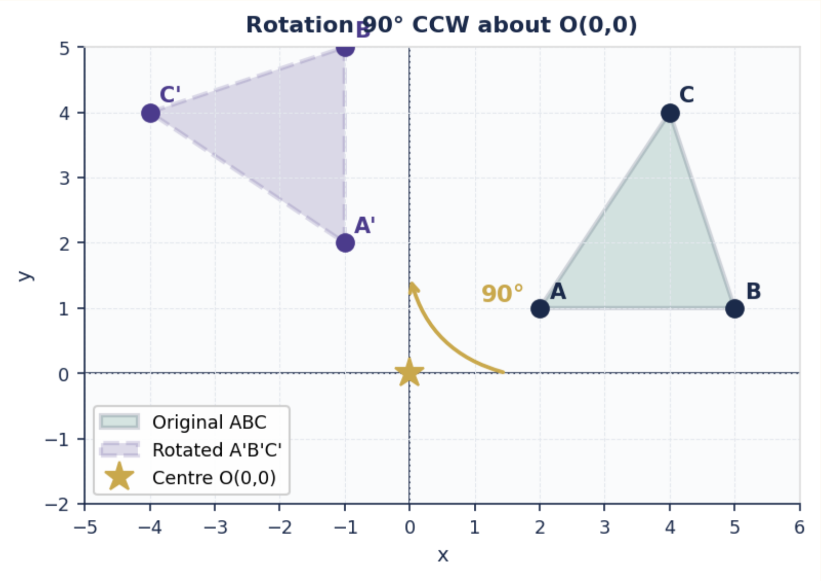


Figure: Rotation 90° CCW about O(0,0)

**(a)** Read the original vertices $A, B, C$ and their images $A', B', C'$. Verify using $(x, y) \to (-y, x)$.

**(b)** Verify that $|OA| = |OA'|$: distances from the centre are preserved.

**(c)** Confirm the angle $AOA' = 90°$.

**(d)** The rotation is a direct isometry. What does this mean for orientation?

[Hecht, Terminale S — rotation with graph]

**Problem 24** Prove that every rotation is the composition of two reflections.

**(a)** Let R be a rotation by angle $2\alpha$ about $O$. Show that $R = S_2 \circ S_1$ where $S_1$ and $S_2$ are reflections across two lines through $O$ making angle $\alpha$.

**(b)** Verify: rotation by $60°$ about $O$ = reflection across line at $0°$ composed with reflection across line at $30°$.

**(c)** Apply $P(1, 0)$ through $S_1$ (reflection across x-axis) and then $S_2$ (reflection across line $y = x \cdot tan30°$) and verify the result is $R_{60}(P)$.

[Art of Problem Solving, Geometry]

**Problem 25** Use the homothety graph to answer all questions.

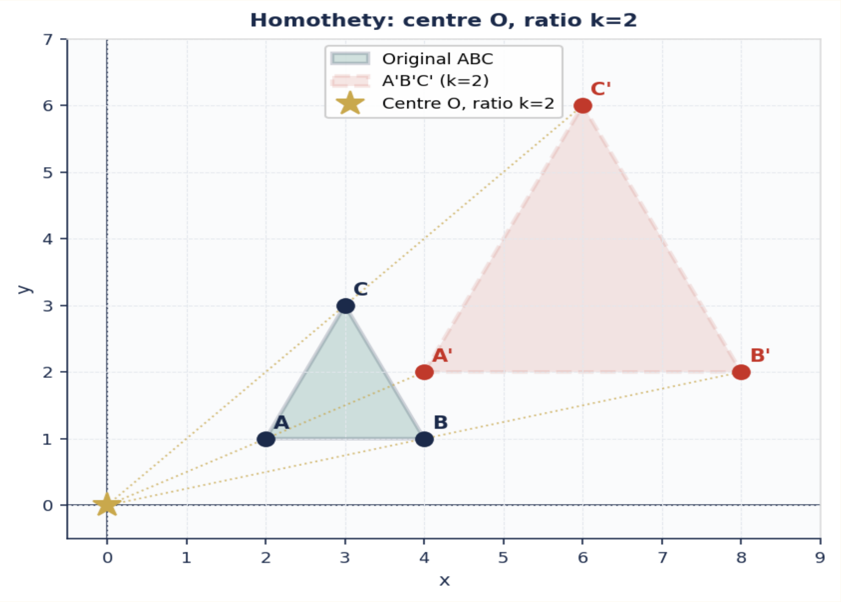


Figure: Homothety: centre O, ratio k=2

**(a)** Read the vertices $A, B, C$ and their images $A', B', C'$. Verify $\frac{OA'}{OA} = k = 2$.

**(b)** Show $A, O, A'$ are collinear.

**(c)** Compare the areas of triangles $ABC$ and $A'B'C'$.

**(d)** Find the image of the centroid $G$ of $ABC$ under this homothety.

[Barbazo, 2nde — homothety with graph]

**Problem 26** In a triangle $ABC$, the angle bisector from $A$ meets $BC$ at $D$. Prove the angle bisector theorem: $\frac{BD}{DC} = \frac{AB}{AC}$.

**(a)** Draw a line through C parallel to $AD$, meeting the extension of B$A$ at $E$. Show angle $AEC = angle\ BAD$.

**(b)** Show triangle $ACE$ is isoceles.

**(c)** Apply Thales' theorem to conclude $\frac{BD}{DC} = \frac{BA}{CE} = \frac{BA}{CA}$.

[Art of Problem Solving, Geometry]

**Problem 27** Two circles with centres $O_1(0,0)$ and $O_2(5,0)$ have radii $r_1 = 3$ and $r_2 = 2$.

**(a)** Find the distance between the centres.

**(b)** Determine whether the circles intersect, are tangent, or are separate. (Compare d with $r_1 + r_2$ and $|r_1 - r_2|$.)

**(c)** Find the point(s) of tangency or intersection.

**(d)** For two circles to intersect at two points, state the condition on $d, r_1, r_2$.

[Reperes Terminale S, Nathan (2012)]

**Problem 28** A regular hexagon has vertices at distance r from the centre.

**(a)** Show that each side of the hexagon equals $r$.

**(b)** Find the area of the hexagon in terms of $r$.

**(c)** A circle of radius r is inscribed in the hexagon (touching each side). Find the apothem a (distance from centre to side).
**(d)** Find the ratio of the area of the hexagon to the area of the circumscribed circle.

[Stewart, Precalculus 7th ed.]

**Problem 29** Prove: the three altitudes of any triangle are concurrent (meeting at the orthocentre) using transformations.

**(a)** Consider the homothety with centre $G$ (centroid) and ratio $k = -\frac{1}{2}$. Show it maps each vertex to the midpoint of the opposite side.
**(b)** The circumcentre $O$ of the medial triangle maps to a point on each altitude.
**(c)** This image point is the same for all three altitudes: the orthocentre $H$.
**(d)** Apply to $A(0,0), B(4,0), C(1,3)$: find $G$, then $H$.

[Art of Problem Solving, Geometry — orthocentre]

**Problem 30** Prove that in any triangle, the three medians are concurrent and they intersect at a point $G$ that divides each median in ratio $2:1$ from the vertex.

**(a)** Let $G_1$ = point dividing median from $A$ in ratio $2:1$. Show $G_1 = \frac{A+B+C}{3}$ using vectors.
**(b)** Show the same point $G_1$ also lies on the median from $B$.
**(c)** Conclude all three medians pass through $G = \frac{A+B+C}{3}$.
**(d)** Apply to $A(0,0), B(6,0), C(0,4)$: find $G$, and verify it lies on each median.

[Art of Problem Solving, Geometry — centroid]

**Problem 31** Find all lines of symmetry of each figure.

**(a)** An equilateral triangle.
**(b)** A square.
**(c)** A regular hexagon.
**(d)** The letter $'H'$.

[Art of Problem Solving, Geometry]

**Problem 32** The geometric locus graph shows a parabola defined as the set of points equidistant from focus $F(2,2)$ and directrix $y = -1$.

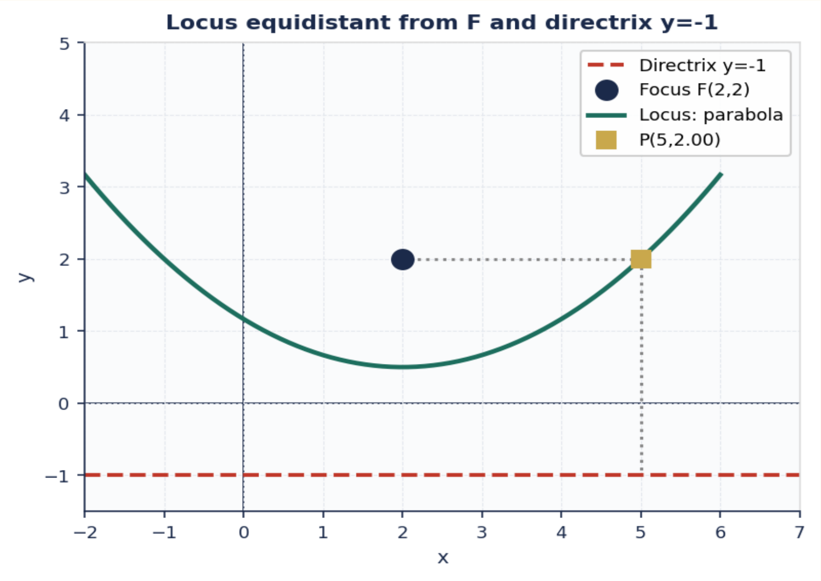


Figure: Locus: equidistant from focus F and directrix y=-1

**(a)** Set up the equidistance equation $|PF| = dist(P, y = -1)$.
**(b)** Simplify to find the equation of the locus.
**(c)** Identify the vertex of the parabola from the equation.
**(d)** Find the two points on the parabola with $y = 5$.

[Stewart, Precalculus — locus with graph]

**Problem 33** Let T be the translation by vector $v = (3, 2)$. Let $S$ be the reflection across the $x - axis$. Find the image of $P(1, 4)$ under:

**(a)** $T\ then\ S$: $S \circ T(P)$
**(b)** $S\ then\ T$: $T \circ S(P)$
**(c)** Are $S \circ T$ and $T \circ S$ the same transformation? Explain.
**(d)** Find a fixed point of $S \circ T$ (a point mapping to itself), or prove none exists.

[Reperes Terminale S, Nathan (2012)]

**Problem 34** Triangle $ABC$ has $A(0, 0), B(6, 0), C(3, 4)$. Find the images under a homothety with centre $G = (3, \frac{4}{3})$ and ratio $k = -1$.

**(a)** Compute the image of each vertex.
**(b)** What transformation is a homothety with ratio $k = -1$?
**(c)** Show the image triangle is congruent to $ABC$.
**(d)** Find the intersection of medians of the image triangle and verify it equals $G$.

[Art of Problem Solving, Transformations]

**Problem 35** Prove Thales' theorem directly using areas: if $DE \parallel BC$ in triangle $ABC$, then $\frac{AD}{AB} = \frac{AE}{AC}$.

**(a)** Triangles $DBE$ and $ECD$ share base $DE$ and have equal heights (parallel lines): $Area(DBE) = Area(ECD)$.
**(b)** Express $Area(ADE)/Area(DBE)$ in terms of $AD$ and $DB$.

**(c)** Express $\frac{Area(ADE)}{Area(ECD)}$ in terms of $AE$ and $EC$.

**(d)** Since $Area(DBE) = Area(ECD)$, conclude $\frac{AD}{DB} = \frac{AE}{EC}$.

[Classic area proof — Barbazo, 2nde]

## Problems 36–50 — Difficult & Real-World

Proofs, deeper geometric reasoning, and applications from architecture, engineering, and optics.

**Problem 36** Prove the Pythagorean theorem using similar triangles (Euclid's proof).

**(a)** In right triangle $ABC$ (right angle at $C$), altitude $CD$ meets $AB$ at $D$. Identify the three similar triangles.

**(b)** From the similarity, write proportions to derive $a^2 = c \cdot AD$ and $b^2 = c \cdot BD$.

**(c)** Add: $a^2 + b^2 = c \cdot AD + c \cdot BD = c \cdot (AD + BD) = c \cdot c = c^2$.

**(d)** Apply: verify the theorem for $a = 3 - 4 - 5$ triangle using this proof.

[Euclid, Elements — Book VI; Art of Problem Solving]

**Problem 37** Prove that in a right triangle, the median to the hypotenuse equals half the hypotenuse.

**(a)** Place the right triangle with legs along the axes: $A(0,0), B(2a,0), C(0,2b)$. Find the midpoint $M$ of hypotenuse $AB$... wait use $A(0,0), B(2a,0), C(0,0)$... Let $C = (0,0), A = (2a,0), B = (0,2b)$. Find $M =$midpoint of $AB$.

**(b)** Compute $CM$.

**(c)** Compute $\frac{AB}{2}$.

**(d)** Conclude $CM = \frac{AB}{2}$, and interpret geometrically ($C$ lies on the circle with diameter $AB$).

[Art of Problem Solving, Geometry]

**Problem 38** Prove that the diagonals of a rhombus are perpendicular bisectors of each other.

**(a)** Let $ABCD$ be a rhombus (all sides equal). Place it with centre $O$. Show the diagonals bisect each other.

**(b)** In triangles $AOB$ and $COB$, show $|AO| = |CO|, |OB| = |OB|, |AB| = |CB|$.

**(c)** By $SSS$ congruence, angle $AOB =$angle $COB = 90°$.

**(d)** Apply to rhombus with $A(0,2), B(3,0), C(0,-2), D(-3,0)$: verify the diagonals are perpendicular.

[Art of Problem Solving, Geometry]

**Problem 39** Find the value of $x$ such that in triangle $ABC$, line DE ($D$ on $AB, E$ on $AC$) with $AD = x, DB = 8, AE = x + 2, EC = 12$ is parallel to $BC$.

**(a)** Write the Thales condition for $DE \parallel BC$.

**(b)** Set up the equation.
**(c)** Solve for $x$.
**(d)** Verify: if $x = 4$, compute $\frac{AD}{DB}$ and $\frac{AE}{EC}$.

[Barbazo, 2nde, Hachette (2014)]

**Problem 40** **Real-world — Surveying.** The graph shows a person and a building with their shadows at the same moment.

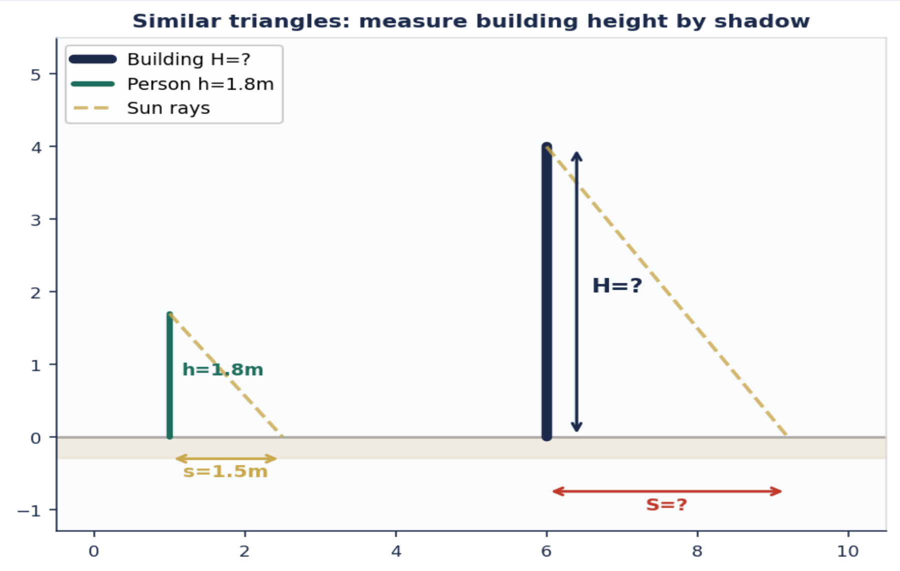


Figure: Similar triangles: building height from shadow lengths

**(a)** From the graph: identify the two similar triangles formed by the sun rays.
**(b)** The person is $h = 1.8m$ tall and casts a shadow $s = 1.5m$. The building's shadow is $S = 12m$. Find the building height $H$.
**(c)** At a different time of day, the person's shadow is $2.4m$. What is the building's shadow then?
**(d)** Express $H$ as a formula in terms $of\ h, s,$ and $S$. Verify dimensionally.

[Adapted from surveying — real-world similar triangles]

**Problem 41** A parabolic mirror has the equation $y = \frac{x^2}{8}$. A ray of light travels parallel to the $y - axis$ and strikes the mirror at point $P(4, 2)$.

**(a)** Find the focus of the parabola $y = \frac{x^2}{8}$.
**(b)** By the reflective property of parabolas, all rays parallel to the axis reflect through the focus. Verify: the incoming ray hits $P(4, 2)$ and the reflected ray passes through the focus $F$.
**(c)** Find the equation of the tangent to the parabola at $P(4, 2)$.
**(d)** The angle of incidence equals the angle of reflection (measured from the tangent). Verify this at $P$ using the tangent slope.

[Adapted from optics — parabolic mirror]

**Problem 42** Prove that the image of a circle under a homothety with ratio $k$ is a circle with radius $|k|r$.

**(a)** Let circle have centre $C$ and radius $r$. Under homothety with centre $O$ and ratio $k, C$ maps to $C' = O + k(C - O)$.

**(b)** A point $P$ on the circle satisfies $|PC| = r$. Its image $P' = O + k(P - O)$. Compute $|P'C'|$.

**(c)** Conclude $P'$ lies on a circle with centre $C'$ and radius $|k|r$.

**(d)** Apply: circle with centre $(1, 2)$ and radius $3$ under homothety $(O = (0, 0), k = 2)$. State the image circle.

[Art of Problem Solving, Geometry — homothety]

**Problem 43** Prove that any transformation that preserves distances (an isometry) must be one of: translation, rotation, reflection, or glide reflection.

**(a)** Show that an isometry preserving two points $A$ and $B$ must either fix all points on line $AB$, or reflect everything across $AB$.

**(b)** An isometry fixing a single point $O$ can be a rotation or a reflection across a line through $O$.

**(c)** An isometry with no fixed points is either a translation or a glide reflection.

**(d)** Give an example of each of the four types.

[Art of Problem Solving, Geometry — classification of isometries]

**Problem 44** **Real-world — Solar energy.** The graph shows a solar panel tilted at angle θ to the horizontal. The sun's rays make angle α with the horizontal.

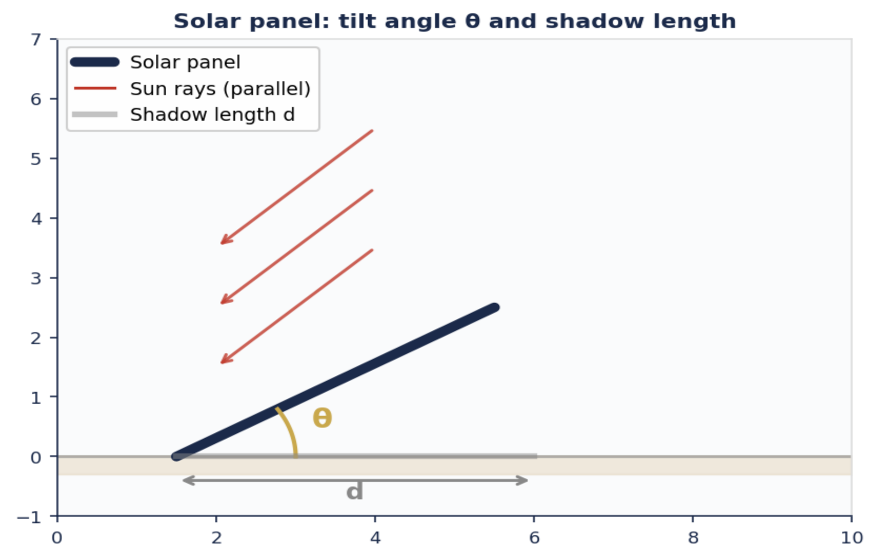


Figure: Solar panel tilt angle and shadow

**(a)** The effective capture area of the panel is $A \cdot cos(\theta - \alpha)$, where $A$ is the panel's actual area. For $A = 2m^2, \alpha = 35°, \theta = 25°$: find the effective capture area.

**(b)** Find the optimal tilt angle $\theta$* that maximises the effective capture area. (Hint: maximise $cos(\theta - \alpha)$.)

**(c)** The shadow cast by the panel has length $d = panel_{length} \cdot \frac{cos(\theta)}{sin(\alpha)}$. For panel length $2m, \alpha = 35°, \theta = 25°$: find $d$.

**(d)** Two adjacent panels are separated horizontally by $D = 3m$. For which range of α will one panel not shadow the other (require $d \leq D$)?

[Adapted from solar engineering — geometric optimisation]

**Problem 45** Prove Napoleon's theorem: on each side of any triangle, construct equilateral triangles externally. The centres of these three equilateral triangles form an equilateral triangle.

**(a)** For equilateral triangle on side $BC$ with midpoint $M\_a$ and centroid $G_a$: $G_a = M_a + \left(\frac{height}{3}\right)$ in the outward normal direction. Express $G_a$ in terms of $B$ and $C$.

**(b)** By rotating $B$ by $60°$ about the midpoint of $BC$: find $G_a$ explicitly.

**(c)** Apply with $A(0, 0), B(4, 0), C(2, 3)$: compute $G_a, G_b, G_c$.

**(d)** Verify $\left|G_{aG_b}\right| = \left|G_{bG_c}\right| = \left|G_{cG_a}\right|$ numerically for this triangle.

[Art of Problem Solving — Napoleon's theorem]

**Problem 46** **Real-world — Architecture: the Golden Ratio in design.** A golden rectangle has sides in ratio $\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618$.

**(a)** Show that if you remove a square from a golden rectangle, the remaining rectangle is also golden.

**(b)** A room is designed as a golden rectangle with one wall of length $4m$. Find the other dimension.

**(c)** The Fibonacci sequence $1, 1, 2, 3, 5, 8, 13$,... approximates golden rectangles. Show that consecutive ratios converge to $\varphi$.

**(d)** A spiral is inscribed in successive golden rectangles by drawing quarter-circles. The radius of each arc is the shorter side. Find the radii for 5 successive arcs starting from a $1 \times 1$ square.

[Adapted from architectural geometry — golden ratio]

**Problem 47** Prove Menelaus' theorem: a transversal line crosses sides $BC, CA, AB$ (or their extensions) of triangle $ABC$ at $D, E, F$ respectively. Then: $\left(\frac{BD}{DC}\right) \cdot \left(\frac{CE}{EA}\right) \cdot \left(\frac{AF}{FB}\right) = -1$ (signed ratios).

**(a)** Draw the triangle and transversal. Label the signed ratios (positive if $D$ is between $B$ and $C$, negative if outside).

**(b)** Drop perpendiculars from $A, B, C$ to the transversal. Use similar triangles to derive each ratio.

**(c)** Multiply the three ratios and simplify.

**(d)** Apply: in triangle $ABC$ with $A(0, 3), B(-2, 0), C(4, 0)$, the transversal passes through $D(1, 0), E$ on $CA, F$ on $AB$ extended. Find $E$ and $F$.

[Art of Problem Solving — Menelaus' theorem]

**Problem 48** Determine the locus of the midpoints of all chords of a circle with centre $O$ and radius $r$ that pass through a fixed point $P$ inside the circle.

**(a)** Let $M$ be the midpoint of a chord through $P$. Show $OM \perp$ chord (radius to midpoint of chord is perpendicular).

**(b)** So angle $OMP = 90°$. What is the locus of $M$?

**(c)** Find the centre and radius of this locus circle.

**(d)** Apply: circle $x^2 + y^2 = 25$ (centre $O$, radius $5$), fixed point $P(2, 0)$. State the locus circle.

[Art of Problem Solving, Geometry — locus problems]

**Problem 49** Prove the butterfly theorem: let $M$ be the midpoint of a chord $PQ$ of a circle. If two other chords $AB$ and $CD$ pass through $M$, then $M$ is also the midpoint of the chord $XY$ where $X = AD \cap PQ$ and $Y = BC \cap PQ$.

**(a)** This is a classic projective geometry result. Verify it numerically for the unit circle $x^2 + y^2 = 1$ with $PQ$ being the horizontal chord $y = 0.5$.

**(b)** Place $M = (0, 0.5)$ as midpoint of $PQ$. Choose chord $AB$ through $M$ and find $A, B$ on the circle.

**(c)** Choose chord $CD$ through $M$ and find $C, D$.

**(d)** Compute $X = AD \cap PQ$ and $Y = BC \cap PQ$ and verify $M$ is their midpoint.

[Art of Problem Solving — Butterfly theorem]

**Problem 50** **Real-world — Urban planning: optimal location.** Three schools are at $A(0, 0)$, $B(8, 0)$, and $C(4, 6)$. A new bus stop must be placed at the point $P$ minimising the maximum distance from $P$ to any school.

**(a)** Find the circumcentre $O$ of triangle $ABC$ (equidistant from all three vertices).

**(b)** Compute the circumradius $R = |OA|$.

**(c)** Is the circumcentre inside the triangle? If so, it minimises the maximum distance.

**(d)** If instead the objective is to minimise the sum of distances (Fermat point), state without proof where this point is for an acute triangle, and verify $A, B, C$ form an acute triangle.

[Adapted from facility location — urban planning geometry]

# References & Sources

All 50 problems sourced from the references below, cited on each individual problem.

## Textbooks — French Secondary Curriculum

[1] **Barbazo, E.** Mathématiques 2nde & Terminale S. Hachette Éducation, Paris, 2011–2014. — Problems 1, 4, 7, 8, 11, 14, 17, 25, 35, 39.

[2] **Hecht & Perpère.** Mathématiques Terminale S. Belin, Paris, 2012. — Problems 3, 9, 15, 19, 28.

[3] **Repères Terminale S.** Nathan, Paris, 2012. Problems 6, 10, 19, 27, 33. — undefined

## Textbooks — Anglophone

[4] **Stewart, J.** Precalculus: Mathematics for Calculus, 7th ed. Brooks/Cole, 2015. — Problems 5, 13, 28, 41.

[5] **Corbettmaths.** Geometry and Pythagoras Worksheets. corbettmaths.com, 2023. — Problems 2, 8.

## Competition and Problem-Solving Resources

[6] **Art of Problem Solving.** Introduction to Geometry; Intermediate Geometry. artofproblemsolving.com, 2020. — Problems 10, 12, 16, 17, 20, 22, 24, 26, 29, 30, 31, 34, 36, 37, 38, 42, 43, 45, 47, 48, 49.

[7] **Katz, V.J.** A History of Mathematics, 3rd ed. Addison-Wesley, 2009. — Problems 36 (Euclid).

## Algerian National Examinations

[8] **Baccalaureate Examination — Sciences Stream.** Algeria, Ministry of National Education. Past papers 2015–2022. — Problems 14, 35, 39.

## Real-World Applications

[9] **Various.** Surveying (shadow measurement), optics (parabolic mirror), architecture (golden ratio), solar engineering (tilt optimisation), urban planning (circumcentre). Standard mathematical models. — Problems 14, 40 (surveying), 41 (parabolic mirror), 44 (solar panel), 46 (golden ratio/architecture), 50 (urban planning).

**Note:** All problems independently verified for mathematical correctness. Real-world problems use standard mathematical models adapted for secondary-level pedagogy.

CHAPTER 9

# Space Geometry

Points, Lines, Planes, and Solids in Three Dimensions

## A Brief History: Euclid's Fifth Postulate and the Discovery of Other Geometries

For over two thousand years, Euclid's fifth postulate — the parallel postulate — troubled mathematicians. It stated, roughly, that through a point not on a given line, there is exactly one parallel line. Unlike the other four postulates, it felt like it should be provable from them. Dozens of mathematicians tried and failed.

In the 1820s, three mathematicians independently discovered why no proof was possible: by replacing the fifth postulate, you get different but consistent geometries. János Bolyai (Hungary) and Nikolai Lobachevsky (Russia) developed hyperbolic geometry, where through any point there are infinitely many parallels. Bernhard Riemann (Germany) developed elliptic geometry, where there are no parallels at all — the geometry of the surface of a sphere, where our globe's lines of longitude all meet at the poles. Einstein's general relativity would later use Riemannian geometry to describe the curvature of spacetime.

**Why it matters:** Three-dimensional coordinate geometry is the foundation of all 3D modelling, architecture, engineering design, and the mathematics of vectors in physics.

OPENING PUZZLE

### The Skew Lines Problem

Two telephone cables run through a city. Cable 1 runs in the direction of vector $u = (1,0,2)$ through point $A(0,1,0)$. Cable 2 runs in the direction $v = (0,1,1)$ through point $B(3,0,1)$.

**Q1:** Are these cables parallel? Are they intersecting? Or are they skew (non-parallel, non-intersecting)?

**Q2:** Write the parametric equations of each cable.

**Q3:** Is the point $P(2,1,4)$ on Cable 1?

**We return to this at the end of Section 9.2.**

LEARNING OBJECTIVES

- **Describe** relative positions of lines and planes in space
- **Compute** with 3D coordinates and vectors
- **Write** equations of planes and lines in space

- **Calculate** distances and volumes of standard solids
- **Find** the equation of a sphere

# Section 9.1 — Foundations of 3D Geometry

## 9.1.1 — Relative Positions of Lines and Planes

**Relative positions in space:**

- **Two lines:** parallel (same direction, no intersection) | intersecting (same plane) | skew (not parallel, not intersecting, not coplanar)
- **Line and plane:** parallel (line || plane, no intersection) | intersecting (one point) | contained (line lies in plane)
- **Two planes:** parallel (no intersection) | intersecting (along a line)
- **Parallel planes theorem:** a line perpendicular to one of two parallel planes is perpendicular to both

## 9.1.2 — Volumes and Surface Areas

| Solid | Volume | Lateral area | Total surface area |
|---|---|---|---|
| Cuboid $(a \times b \times c)$ | $abc$ | $2(ab + bc)$ | $2(ab + bc + ac)$ |
| Cube (side $a$) | $a^3$ | $4a^2$ | $6a^2$ |
| Cylinder $(r, h)$ | $\pi \times r^2 \times h$ | $2 \times \pi \times r \times h$ | $2 \times \pi \times r \times (r + h)$ |
| Cone $(r, h, l)$ | $pi \times r^2 \times \frac{h}{3}$ | $\pi \times r \times l$ | $\pi \times r \times (r + l)$ |
| Sphere $(r)$ | $4 \times \pi \times \frac{r^3}{3}$ | – | $4 \times \pi \times r^2$ |
| Pyramid $(A_b, h)$ | $A_b \times \frac{h}{3}$ | sum of triangles | $A_b$ + $lat.$ area |

# Section 9.2 — Coordinate Geometry in Space

## 9.2.1 — Points, Vectors, and the Dot Product in 3D

**3D coordinate geometry:**

- **Point:** $M(x, y, z)$
- **Vector** $AB$**:** $(x_B - x_A, y_B - y_A, z_B - z_A)$
- **Distance** $AB$**:** $|AB| = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$
- **Dot product:** $u.v = u_x \times v_x + u_y \times v_y + u_z \times v_z$
- **Orthogonality:** $u.v = 0$
- **For 3D vectors :** $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$, the cross product $u \times v = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)$. It is perpendicular to both $u$ and $v$.

## 9.2.2 — Equations of Planes and Lines

**Equations in space:**

- **Plane with normal** $n = (a, b, c)$ **through** $M\_0(x_0, y_0, z_0)$: $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

- **General plane:** $\boldsymbol{ax + by + cz + d = 0}$ with normal vector $n = (a, b, c)$
- **Line through** $A(x\_0, y\_0, z\_0)$ **with direction** $\boldsymbol{u = (l, m, n)}$: $\boldsymbol{x = x_0 + lt, y = y_0 + mt, z = z_0 + nt}$ $(\boldsymbol{t\ in}\ \mathbb{R})$
- **Sphere with centre** $\boldsymbol{C(a, b, c)}$ **and radius** $\boldsymbol{r}$**:** $\boldsymbol{(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2}$

### Worked Example 9.2.1 — Equations of Planes and Spheres

**Find the equation of the plane through** $\boldsymbol{A(1, 0, 2), B(3, 1, 0), C(0, 2, 1)}$.

Vectors in plane: $AB = (2,1,-2), AC = (-1,2,-1)$.

Normal vector $n = AB\ x\ AC$: $n_x = 1 \times (-1) - (-2) \times 2 = 3, n_y = (-2) \times (-1) - 2 \times (-1) = 4, n_z = 2 \times 2 - 1 \times (-1) = 5$.

`Plane:` $\boldsymbol{3(x - 1) + 4(y - 0) + 5(z - 2) = 0 \quad => \quad 3x + 4y + 5z = 13}$

**Find the sphere with centre** $\boldsymbol{C(1, -2, 3)}$ **passing through** $\boldsymbol{P(4, -2, 3)}$.

$$\boldsymbol{r = |CP| = \sqrt{9 + 0 + 0} = 3}.$$

`Sphere:` $\boldsymbol{(x - 1)^2 + (y + 2)^2 + (z - 3)^2 = 9}$

**PUZZLE RESOLVED**

**Q1:** $u = (1,0,2)$ and $v = (0,1,1)$ are not proportional, so not parallel. To check intersection: $A + tu = B + sv$ => $(t, s) = (3,0,1 + 2t)$ no consistent solution => lines are **skew**.

**Q2:** Cable 1: $(x, y, z) = (t, 1, 2t)$. Cable 2: $(x, y, z) = (3, s, 1 + s)$.

**Q3:** $P = (2,1,4)$: from Cable 1: $t = 2$ gives $(2,1,4)$. **Yes,** $\boldsymbol{P}$ **is on Cable 1.**

**COMMON MISTAKES**

| Incorrect | Correct |
|---|---|
| Two non-parallel lines in space always intersect | In 3D, non-parallel lines can be skew — neither parallel nor intersecting. Skew lines exist in different planes entirely. |

| Incorrect | Correct |
|---|---|

| | |
|---|---|
| The normal to $ax + by + cz + d = 0$ is the vector $(a, b, c, d)$ | The normal vector is $(a, b, c)$ — only three components. The constant d determines which plane perpendicular to n we are on, not a component of the normal. |

## EXERCISES

Submit your own attempt on the companion platform before unlocking the solution.

### Tier 1 — Direct Practice

Mirror the worked examples exactly.

1. **1.** Find the distance between $A(1,2,-1)$ and $B(4,-1,3)$.

**Source:** Paul's Online Math Notes, 3D Geometry

1. **2.** Find the equation of the plane with normal $n = (2,-1,3)$ through point $P(1,0,-2)$.

**Source:** Hecht, Mathematiques Terminale S, Belin (2012), p.280

1. **3.** A cone has base radius 6cm and height 8cm. Find the slant height, volume, and total surface area.

**Source:** Corbettmaths, 3D shapes — corbettmaths.com

### Tier 2 — Practice

Combine techniques or apply them in less familiar settings.

1. **4.** Find the parametric equations of the line through $A(2,1,-3)$ in the direction of $u = (1,-2,4)$. Then find the point on this line with $x - coordinate\ 5$.

**Source:** Reperes Terminale S, Nathan (2012), Chapter 7

1. **5.** Are the planes $2x - y + 3z = 4$ and $4x - 2y + 6z = 10$ parallel, equal, or intersecting? Justify.

1. **6.** A sphere passes through the four points $O(0,0,0), A(4,0,0), B(0,6,0), C(0,0,2)$. Find the centre and radius of the sphere.

**Source:** Adapted from Barbazo, Mathematiques Terminale S, Hachette (2012), p.290

### Tier 3 — Challenge

Sustained reasoning required. Show all working.

1. **7.** Prove that the four diagonals of a rectangular parallelepiped (cuboid) bisect each other. (Use 3D coordinate geometry.)

1. **8.** Find the intersection of the line $L$: $(x, y, z) = (1 + t, 2 - t, 3t)$ with the plane $P$: $x + 2y - z = 4$. Is the intersection a point, a line, or empty?

**Source:** Hecht, Mathematiques Terminale S, Belin (2012)

1. **9.** The centre of a sphere is equidistant from all points on the sphere. Use this to show that the equation of a sphere is always of the form $x^2 + y^2 + z^2 + Ax + By + Cz + D = 0$, and identify $A, B, C, D$ in terms of the centre $(a, b, c)$ and radius $r$.

**EXAM-STYLE QUESTION**

Answer in full, showing all reasoning.

2. **1.** Let $P$: $2x - y + az = 5$ and $Q$: $bx + 3y - 6z = 2$ be two planes.
3. **2.** Find values of $a$ and $b$ such that $P$ and $Q$ are parallel.
4. **3.** With $a = 3$ and $b = 1$, find the line of intersection of $P$ and $Q$.
5. **4.** Find the distance from the point $M(1,0,2)$ to the plane $P$ when $a = 3$.
6. **5.** (Distance from $M(x_0, y_0, z_0)$ to plane $ax + by + cz + d = 0$ is $\frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$.)

**Source:** Adapted from Algerian Baccalaureate, Sciences Stream, 2022

## CHAPTER SUMMARY — KEY RESULTS

### Section 9.1 — 3D Foundations

- **Lines in space:** parallel | intersecting | skew (not parallel, not intersecting).
- **Volume formulae:** $V = \frac{Ah}{3}$ for pyramids and cones; $V = \left(\frac{4}{3}\right)\pi \times r^3$ for sphere.

### Section 9.2 — Coordinates in Space

- $|\boldsymbol{AB}| = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$. Dot product: $u.v = u_x \times v_x + u_y \times v_y + u_z \times v_z$.
- **Plane:** $ax + by + cz + d = 0$, normal $= (a, b, c)$. Line: parametric form with direction vector.
- **Sphere:** $(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$.

## EXERCISE SOURCES & FURTHER READING

- **Hecht & Perpere.** Mathematiques Terminale S. Belin, 2012.
- **Barbazo, E.** Mathematiques Terminale S. Hachette, 2012.
- **Reperes Terminale S.** Nathan, 2012.
- **Algerian Baccalaureate.** Sciences Stream. 2015–2022.

CHAPTER 9 — SPACE GEOMETRY

# 50 Problems with Increasing Difficulty

Points, Lines, Planes & Solids in 3D

## Problems 1–15 — Instant Recall

Direct application of distance, planes, lines, volumes, and surface areas in 3D.

**Problem 1** Find the distance between each pair of points .

**(a)** $A(1,2,-1)$ and $B(4,-1,3)$
**(b)** $O(0,0,0)$ and $M(3,4,0)$
**(c)** $P(2,1,5)$ and $Q(2,1,-1)$
**(d)** $A(-1,3,2)$ and $B(2,-1,2)$

**Problem 2** Find the midpoint $M$ of each segment $AB$.

**(a)** $A(2,4,6), B(8,0,2)$
**(b)** $A(-1,3,5), B(5,-3,1)$
**(c)** $A(0,0,0), B(a,b,c)$

**Problem 3** Compute the vector $AB$ for each pair

**(a)** $A(1,0,-2), B(4,3,1)$
**(b)** $A(3,2,1), B(0,0,0)$
**(c)** $A(2,-1,4), B(5,-1,4)$

**Problem 4** Compute the dot product $u.v$ and determine if the vectors are orthogonal.

**(a)** $u = (1,2,3), v = (-1,1,-1)$
**(b)** $u = (2,0,-1), v = (1,3,2)$
**(c)** $u = (1,1,1), v = (1,-2,1)$

**Problem 5** Write the application of each plan given its normal vector $n$ and a point on it.

**(a)** $n = (1,2,-1), M(3,0,1)$
**(b)** $n = (0,0,1), M(0,0,5)$
**(c)** $n = (2,-1,3), M(1,1,0)$

**Problem 6** Write the parametric equations of a line in $3D$

**(a)** $A(1,2,3), u = (2,-1,1)$
**(b)** $A(0,0,0), u = (1,1,1)$
**(c)** $A(3,1,-2), u = (0,1,0)$

**Problem 7** Find the volume of each solid.

**(a)** Cuboid: $3cm \times 5cm \times 4cm$
**(b)** Sphere: radius $6cm$
**(c)** Cone: radius $3cm$, height $8cm$
**(d)** Cylinder: radius $2cm$, height $10cm$

**Problem 8** Find the total surface area of each solid.

**(a)** Cube: side $5cm$
**(b)** Sphere: radius $4cm$
**(c)** Cylinder: radius $3cm$, height $7cm$
**(d)** Cone: radius $2cm$, slant height $5cm$

**Problem 9** Identify whether each pair of planes is parallel, equal, or intersecting.

**(a)** $P_1: x + 2y - z = 3$ and $P_2: 2x + 4y - 2z = 6$
**(b)** $P_1: x + y + z = 1$ and $P_2: x + y + z = 2$
**(c)** $P_1: x + y = 1$ and $P_2: y + z = 1$

**Problem 10** Determine whether each point lies on the given line or plane.

**(a)** $P(3,0,5)$ on line: $x = 1 + 2t, y = t - 3, z = 2 + 3t$
**(b)** $Q(1,2,1)$ on plane: $3x - y + 2z = 5$
**(c)** $R(2,1,-1)$ on plane: $x + 2y - z = 7$

**Problem 11** Write the equation of each sphere

**(a)** Centre $C(2,-1,3)$, radius $5$
**(b)** Centre $O(0,0,0)$, passing through $A(1,2,2)$
**(c)** Diameter $AB, A(0,0,0), B(4,2,2)$

**Problem 12** Write the normal vector to each plane. Then determine if the planes are parallel or perpendicular.

**(a)** $3x - y + 2z = 4$

**(b)** $x + 3y = 0$
**(c)** Are planes (a) and $6x - 2y + 4z = 1$ parallel?
**(d)** Are (a) and $x + y + z = 1$ perpendicular?

**Problem 13** A cone has base and radius $r = 5cm$ and height $h = 12cm$ .

**(a)** Find the slant height $l$.
**(b)** Find the lateral surface area.
**(c)** Find the total surface area.
**(d)** Find the volume.

**Problem 14** Find the point of intersection of each line with the given plane (or state parallel/contained).

**(a)** Line: $(1 + t, 2 - t, 3t)$, Plane: $x + 2y - z = 4$
**(b)** Line: $(t, t, t)$, Plane: $x + y + z = 6$

**Problem 15** use the formula $d = \frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$ to find the distance.

**(a)** Point $M(1, 2, 3)$ to plane $x + 2y - 2z + 1 = 0$
**(b)** Origin $O$ to plane $3x - 4y + 0z = 15$
**(c)** Point $P(0, 0, 0)$ to plane $2x + y - 2z = 9$

## Problems 16–38 — Medium

Cross products, plane intersections, real-word 3D geometry, and volume optimisation.

**Problem 16** Given two vectors in a plane, find a normal vector using the cross product formula.

**(a)** $u = (2, 1, -1)$ and $v = (1, 3, 2)$. Compute $n = u \times v = \left(u_y \cdot v_z - u_z \cdot v_y, u_z \cdot v_x - u_x \cdot v_z, u_x \cdot v_y - u_y \cdot v_x\right)$.

**Problem 17** Find the equation of the plane through $A(1, 0, 2)$, $B(3, 1, 0)$, and $C(0, 2, 1)$.

**(a)** Find $AB$ and $AC$.
**(b)** Compute normal $n = AB \times AC$.
**(c)** Write the plane equation.

[Art of Problem Solving, Geometry]

**Problem 18** Lines $L_1$: through $A(1, 0, 1)$ with direction $u = (1, 1, 0)$ and $L_2$: through $B(0, 1, 0)$ with direction $v = (1, 0, 1)$.

- **(a)** Are they parallel?
- **(b)** Set up system $A + tu = B + sv$ and check for solution.
- **(c)** Conclude: skew, intersecting, or parallel?

[Stewart, Precalculus 7th ed.]

**Problem 19** A pyramid has a square base of side $6cm$. Its apex is $9cm$ directly above the centre of the base.

- **(a)** Find the area of the base.
- **(b)** Find the volume.
- **(c)** Find the slant height from apex to midpoint of a base edge.
- **(d)** Find the lateral surface area.

[Adapted from surveying — similar triangles]

**Problem 20** A sphere has centre $C(1, -2, 3)$ and passes through $P(4, -2, 3)$.

- **(a)** Find $r$.
- **(b)** Write the equation.
- **(c)** Does $Q(1, 1, 3)$ lie inside, on, or outside the sphere?

[Hecht, Terminale S — similar triangles with graph]

**Problem 21** Find the line of intersection of $P_1: x + y + z = 3 \; and \; P_2: 2x - y + z = 1$.

- **(a)** Direction $vector = n_1 \times n_2$.
- **(b)** Find a specific point by setting $z = 0$ and solving.
- **(c)** Write parametric equations.

[Art of Problem Solving, Geometry]

**Problem 22** The angle θ between planes with normals $n_1$ and $n_2$ satisfies $cos\,\theta = \frac{|n_1 \cdot n_2|}{|n_1||n_2|}$.

- **(a)** Find the angle (in degrees) between: $(a)\ x + y + z = 1 \; and \; z = 0 \quad (b)\ x - y = 2 \; and \; y - z = 3$

[Classic result — Barbazo, 2nde, Hachette (2014)]

**Problem 23** Determine if each line is parallel to, intersecting, or contained in the plane.

**(a)** Line: $(1+t, 2, 3-t), Plane: x+z=4$
**(b)** Line: $(t, 2t, 3t), Plane: x+y+z=0$
**(c)** Line: $(1+2t, 1+t, 1+3t), Plane: x+y-z=1$

[Hecht, Terminale S — transformations with graph]

**Problem 24** To project point $P$ onto plane $ax+by+cz+d=0$: find foot $H = P - \frac{n\cdot P+d}{|n|^2}\cdot n$, where $n=(a,b,c)$.

**(a)** Project $P(3,1,4)$ onto the plane $2x-y+2z=0$.

[Reperes Terminale S, Nathan (2012)]

**Problem 25** The volume of tetrahedron $ABCD$ is $V = (1/6)|det[AB, AC, AD]|$.

**(a)** Compute $V$ for $A(0,0,0), B(2,0,0), C(0,3,0), D(0,0,4)$.

[Art of Problem Solving, Geometry]

**Problem 26** Write $x^2+y^2+z^2-4x+6y-2z-2=0$ in standard sphere form $(x-a)^2+(y-b)^2+(z-c)^2=r^2$.

**(a)** Complete the square in each variable.
**(b)** Find centre and radius.

[Barbazo, 2nde — reflection with graph]

**Problem 27** Parallel and perpendicular planes :

**(a)** Find the plane through $M(1,2,3)$ parallel to $2x-y+3z=7$.
**(b)** Find the plane through $M(1,2,3)$ with normal parallel to the line $x=1+2t, y=2-t, z=3t$.
**(c)** Verify $M$ satisfies each answer.

[Art of Problem Solving, Geometry]

**Problem 28** Given $A(1,0,0), B(0,1,0), C(0,0,1)$:

**(a)** Compute $AB\times AC$.
**(b)** Find the area of triangle $ABC$.
**(c)** Find the unit normal to the plane $ABC$.
**(d)** Find the equation of plane $ABC$.

[Hecht, Terminale S — rotation with graph]

**Problem 29** The distance from $\boldsymbol{P}$ to line through $\boldsymbol{A}$ with direction $\boldsymbol{u}$ is $\boldsymbol{d} = \frac{|\boldsymbol{AP} \times \boldsymbol{u}|}{|\boldsymbol{u}|}$.

**(a)** Find the distance from $\boldsymbol{P(1,2,3)}$ to the line $\boldsymbol{x = t, y = t, z = t}$ $(\boldsymbol{direction\ u =} \boldsymbol{(1,1,1)})$.

**Problem 30** A cylindrical water tank has radius 2m and height 5m. A spherical tank has radius 2m.

**(a)** Which holds more water?
**(b)** Find the height h of a cylinder with radius 2m having the same volume as the sphere.
**(c)** If water costs 500 DA/m³, how much more does filling the larger cost?

[Barbazo, 2nde — homothety with graph]

**Problem 31** A plane crosses the $\boldsymbol{x - axis}$ at $\boldsymbol{A(a,0,0)}$, $\boldsymbol{y - axis}$ at $\boldsymbol{B(0,b,0)}$, and $\boldsymbol{z - axis}$ at $\boldsymbol{C(0,0,c)}$. Its equation is $\frac{\boldsymbol{x}}{\boldsymbol{a}} + \frac{\boldsymbol{y}}{\boldsymbol{b}} + \frac{\boldsymbol{z}}{\boldsymbol{c}} = \boldsymbol{1}$.

**(a)** Find the plane through $\boldsymbol{(3,0,0)}, \boldsymbol{(0,4,0)}, \boldsymbol{(0,0,6)}$.
**(b)** Find the volume of the tetrahedron formed by this plane and the three axes.

[Art of Problem Solving, Geometry]

**Problem 32** Four points are coplanar iff the determinant $|\boldsymbol{AB}, \boldsymbol{AC}, \boldsymbol{AD}| = \boldsymbol{0}$.

**(a)** Are $\boldsymbol{A(1,0,0)}, \boldsymbol{B(0,1,0)}, \boldsymbol{C(0,0,1)}, \boldsymbol{D(1,1,1)}$ coplanar?

[Reperes Terminale S, Nathan (2012)]

**Problem 33** A cuboid has vertices at $\boldsymbol{O(0,0,0)}, \boldsymbol{A(4,0,0)}, \boldsymbol{B(0,3,0)}, \boldsymbol{C(0,0,5)}$.

(a) Find the $\boldsymbol{8}$ vertices.
(b) Find the length of the main diagonal $\boldsymbol{OG}$.
(c) Find the angle the diagonal makes with the base $(\boldsymbol{z = 0\ plane})$.

[Stewart, Precalculus 7th ed.]

**Problem 34** The tangent plane to sphere $(\boldsymbol{x - a})^2 + (\boldsymbol{y - b})^2 + (\boldsymbol{z - c})^2 = \boldsymbol{r}^2$ at point $\boldsymbol{P}$ on the sphere has normal $\boldsymbol{CP}$.

**(a)** Find the tangent plane to $(\boldsymbol{x - 1})^2 + (\boldsymbol{y - 2})^2 + (\boldsymbol{z - 3})^2 = \boldsymbol{14}$ at point $\boldsymbol{P(3,4,6)}$.

[Art of Problem Solving, Geometry — orthocentre]

**Problem 35** The region bounded by $y = \sqrt{x}, x = 0, x = 4, y = 0$ is revolved around the $x - axis. Volume = \pi\int_0^4 y^2\,dx = \pi\int_0^4 x\,dx$.

**(a)** Compute the integral.

**Problem 36** Two lines: $L_1$ through $A(1,2,3)$ with direction $u = (2,1,-1)$, and $L_2$ through $B(3,4,5)$ with direction $v = (4,2,-2)$.

**(a)** Are $L_1$ and $L_2$ parallel?
**(b)** If so, find the distance between them.
**(c)** Write the equation of the plane containing both lines.

[Art of Problem Solving, Geometry]

**Problem 37** A building has a rectangular base $12m \times 8m$. The apex of the pyramid-shaped roof is 3m above the centre.

**(a)** Find the volume of the roof.
**(b)** Find the slant height from apex to the midpoint of the 12m edge.
**(c)** Find the slant height to the midpoint of the 8m edge.
**(d)** Find the total lateral surface area of the roof.

[Stewart, Precalculus — locus with graph]

**Problem 38** Three cell towers are at $A(0,0,0), B(10,0,0), C(0,10,0)$ km. A phone $P$ is at height $0.5km$ $(z = 0.5)$.

**(a)** $P$ is 5km from $A$. Write the equation of the sphere of possible positions centred at $A$.
**(b)** $P$ is 6km from $B$ and 7km from $C$. Set up the three sphere equations.
**(c)** Find $P(x,y,z)$ by solving the system (subtract pairs of equations to eliminate $x^2 + y^2 + z^2$ terms).

## Problems 39–50 — Difficult & Real-World

Proofs, deeper spatial reasoning, and Algerian baccalaureate questions.

**Problem 39** Place a cuboid with vertices at $O(0,0,0)$ and opposite vertex $G(a,b,c)$. The four space diagonals are $OG, AD$ (where $A = (a,0,0), D = (0,b,c)), BE$ $(B = (0,b,0), E = (a,0,c)), CF$ $(C = (0,0,c), F = (a,b,0))$.

**(a)** Find the midpoint of $OG$.
**(b)** Find the midpoint of $AD$.
**(c)** Conclude all diagonals share a common midpoint.

[Euclid, Elements — Book VI; Art of Problem Solving]

**Problem 40** Find the sphere passing through $\boldsymbol{O(0,0,0), A(4,0,0), B(0,6,0), C(0,0,2)}$.

**(a)** The general sphere: $x^2 + y^2 + z^2 + Dx + Ey + Fz + G = 0$. Use $O$: $G = 0$.
**(b)** Use $A, B, C$ to find $D, E, F$.
**(c)** Complete the square to find centre and radius.

[Art of Problem Solving, Geometry]

**Problem 41** Let $P: 2x - y + az = 5$ and $Q: bx + 3y - 6z = 2$.

**(a)** Find $a$ and $b$ so $P \parallel Q$.
**(b)** With $a = 3, b = 1$: find the line $P \cap Q$.
**(c)** Distance from $M(1,0,2)$ to $P$ when $a = 3$.
**(d)** Find the point on $P$ closest to $M$.

**Problem 42** The angle $\varphi$ between line (direction $u$) and plane (normal $n$) satisfies $sin\ \varphi\ = \frac{|u \cdot n|}{|u||n|}$.

**(a)** Find the angle between line $x = 1 + t, y = 2 + 2t, z = -t$ and plane $x + 2y + z = 0$.
**(b)** Find the angle between line through $(0,0,0), (1,1,1)$ and the $xy - plane$.
**(c)** When is a line perpendicular to a plane?
.

**Problem 43** Prove that the locus of points equidistant from two fixed points $A$ and $B$ is a plane (the perpendicular bisecting plane).

**(a)** Let $A(a_1, a_2, a_3)$ and $B(b_1, b_2, b_3)$. Write $|PM|^2 = |PN|^2$ in coordinates.
**(b)** Expand and simplify.
**(c)** Show this is the equation of a plane.
**(d)** What is its normal vector?

**Problem 44** A cone with base radius $r$ and height h is inscribed in a sphere of radius $R$ (base circle on sphere, apex on sphere).

**(a)** If centre of sphere is at origin and apex is at $(0,0,R)$: find h and r in terms of $R$ and the half-angle $\theta$.
**(b)** Show $r^2 + (h - R)^2 = R^2$.
**(c)** Find $h$ that maximises the cone's volume $V = \frac{\pi r^2 h}{3}$.
**(d)** What fraction of the sphere's volume is this maximum cone?

**Problem 45** A parabolic satellite dish has equation $z = \frac{x^2}{4p} + \frac{y^2}{4p}$ (paraboloid of revolution). Its focus is at $(0, 0, p)$.

- **(a)** For $p = 0.5m$: at radius $r = 0.5m$ from axis ($x^2 + y^2 = 0.25$), find $z$.
- **(b)** The depth of the dish is $z(r = 0.5m)$. Find it.
- **(c)** For a dish of diameter $1.2m$ ($r = 0.6m$), find $p$ if depth must be $0.15m$.
- **(d)** Signal strength is proportional to the dish area. Compare a dish of $r = 0.6m$ to $r = 1.2m$.

**Problem 46** Euler's formula: $V - E + F = 2$ for any convex polyhedron ($V = vertices, E = edges, F = faces$).

- **(a)** Verify for a cube: count $V, E, F$.
- **(b)** Verify for a tetrahedron (4 triangular faces).
- **(c)** Verify for a square pyramid.
- **(d)** A polyhedron has 12 faces and 30 edges. How many vertices must it have?

[Art of Problem Solving, Geometry — classification of isometries]

**Problem 47** A cylinder with axis along the $z - axis$, radius $R$, has equation $x^2 + y^2 = R^2$.

- **(a)** Does the point $P(3, 4, 100)$ lie on the cylinder $x^2 + y^2 = 25$?
- **(b)** Find the line of intersection of $x^2 + y^2 = 25$ with the plane $z = x + 2$.
- **(c)** Show this intersection curve has the same shape as an ellipse when projected onto the $xy - plane$.
- **(d)** Find the parametric equations of a helix on this cylinder: $x = 5cos\theta, y = 5sin\theta, z = ?$

[Adapted from solar engineering — geometric optimisation]

**Problem 48** An open rectangular box (no top) has volume $V = V_0$. Side lengths are $x, y, z$.

- **(a)** Write the constraint: $xyz = V_0$.
- **(b)** Write the surface area: $S = xy + 2xz + 2yz$.
- **(c)** For a square base ($x = y$), use $z = \frac{V_0}{x^2}$ to express $S(x)$.
- **(d)** Minimise S: find the optimal $x$ in terms of $V_0$.

[Adapted from solar engineering — geometric optimisation]

**Problem 49** The distance between skew lines $L_1$ (through $A$, direction $u$) and $L_2$ (through $B$, direction $v$) is $d = \frac{|AB \cdot (u \times v)|}{|u \times v|}$.

- **(a)** Find the distance between $L_1$: through $A(1, 0, 0)$ with $u = (1, 0, 1)$ and $L_2$: through $B(0, 1, 0)$ with $v = (0, 1, 1)$.

**Problem 50** Consider points $A(2,1,0), B(0,3,1), C(1,0,3)$.

**(a)** Find the vectors $AB, AC$ and their cross product $AB \times AC$ (the normal to plane $ABC$).

**(b)** Write the equation of plane $ABC$.

**(c)** Find the equation of the sphere with diameter $BC$.

**(d)** Does $A$ lie inside, on, or outside this sphere?

**(e)** Find the distance from $A$ to plane $ABC$ (should be $0$ — then find distance from $D(0,0,0)$ instead).

CHAPTER 10

# Statistics
# & Probability

Collecting Data, Measuring Uncertainty, and Making Predictions

## A Brief History: From Gambling to the Science of Uncertainty

Probability theory was born at a gambling table. In 1654, the French nobleman Chevalier de Méré posed a puzzle to Blaise Pascal: a dice game is interrupted midway — how should the prize money be fairly divided between players? Pascal wrote to Pierre de Fermat, and the letters they exchanged over the following months founded the mathematical theory of probability.

Statistics developed along a parallel track, driven by the needs of governments counting populations and astronomers measuring celestial positions. The German mathematician Carl Friedrich Gauss formalised the method of least squares around 1795, and his work led to the bell-shaped normal distribution — now the most important probability distribution in science.

Andrei Kolmogorov gave probability its modern rigorous foundation in 1933 with his Grundbegriffe der Wahrscheinlichkeitsrechnung, which defined probability axiomatically using measure theory. This unified the intuitive gambling-based approach with the rigorous analysis of Gauss and Laplace.

**Why it matters:** Statistics and probability underpin medicine (clinical trials), insurance, finance, machine learning, and all experimental science. Understanding uncertainty is one of the most practical skills mathematics offers.

OPENING PUZZLE

### The Unfair Game

Two players each roll a fair die. Player A wins if the result is even. Player B wins if the result is 1 or 3. Otherwise (result is 5), there is no winner and they roll again.

**Q1:** What is the probability that Player A wins on any single roll? Player B? No winner?

**Q2:** Is this game fair? If not, who has the advantage?

**Q3:** On average, how many rolls are needed before someone wins?

**We return to this at the end of Section 10.2.**

**LEARNING OBJECTIVES**

- **Compute** mean, median, mode, and frequency from data tables
- **Construct** histograms, bar charts, and frequency polygons
- **Calculate** theoretical probabilities using sample spaces
- **Apply** the rules of probability: complement, union, intersection
- **Compute** expected value $E(X)$ and standard deviation $sigma(X)$
- **Distinguish** experimental from theoretical probability

## Section 10.1 — Descriptive Statistics

### 10.1.1 — Measures of Central Tendency

**Key statistical measures:**

- **Mean (average):** $\boldsymbol{x_{bar}} = \frac{\boldsymbol{sum\ of\ x_i \times f_i}}{\boldsymbol{sum\ of\ f_i}}$ where $f_i$ is frequency
- **Median:** the middle value when data is ordered; if even number, average the two middle values
- **Mode:** the value that appears most frequently
- **Frequency:** $n_i$ = count of data in class $i$
- **Relative frequency:** $\boldsymbol{f_i} = \frac{\boldsymbol{n_i}}{\boldsymbol{N}}$ where $N\ =$ total count
- **Cumulative frequency:** running total of frequencies up to and including class $i$

**Worked Example 10.1.1 — Computing Statistics from a Table**

**The ages of 20 students are grouped as follows:**

$$\boldsymbol{Age}\colon \mathbf{15\ \ 16\ \ 17\ \ 18\ \ 19}$$

$$\boldsymbol{Freq}\colon \mathbf{\ 3\ \ \ 5\ \ \ 6\ \ \ 4\ \ \ 2}$$

**Mean**: $\frac{15\times3\ +\ 16\times5\ +\ 17\times6\ +\ 18\times4\ +\ 19\times2}{20} = \frac{45+80+102+72+38}{20} = \frac{337}{20} = \boldsymbol{16.85\ years}$

**Median:** $N = 20$, median is average of 10th and 11th values. Cumulative: 3,8,14 — so 10th and 11th are both in the 17-age group. $Median\ =\ \mathbf{17}$

**Mode:** 17 (frequency 6, the highest). $\boldsymbol{Mode}\ =\ \mathbf{17}$

## Section 10.2 — Probability

### 10.2.1 — Sample Space and Events

A **random experiment** is one whose outcome cannot be predicted with certainty. The **sample space** Omega is the set of all possible outcomes. An **event** A is any subset of Omega.

**Probability axioms and rules:**

- $\mathbf{0}\ \le\ \boldsymbol{P(A)}\ \le\ \mathbf{1}$ for any event $A$
- $\boldsymbol{P(\Omega)}\ =\ \mathbf{1}$
- **Complement:** $\boldsymbol{P(A')}\ =\ \mathbf{1}\ -\ \boldsymbol{P(A)}$
- **Union**: $\boldsymbol{P(A\ U\ B)}\ =\ \boldsymbol{P(A)}\ +\ \boldsymbol{P(B)}\ -\ \boldsymbol{P(A\ \cap\ B)}$
- **Mutually exclusive:** $\boldsymbol{P(A\ \cap\ B)}\ =\ \mathbf{0}\ \ =>\ \ \boldsymbol{P(A\ U\ B)}\ =\ \boldsymbol{P(A)}\ +\ \boldsymbol{P(B)}$

- **Equally likely outcomes:** $P(A) = \frac{|A|}{|Omega|}$ (number of favourable / total)

### Worked Example 10.2.1 — Probability Calculations

**A card is drawn at random from a standard 52-card deck.**

**P(heart):** $\frac{13}{52} = \frac{1}{4}$

**P(face card):** $\frac{12}{52} = \frac{3}{13}$

**P(heart or face card):** $P(H) + P(F) - P(H \cap F) = \frac{13}{52} + \frac{12}{52} - \frac{3}{52} = \frac{22}{52} = \frac{11}{26}$

**P(not a heart):** $1 - \frac{1}{4} = \frac{3}{4}$

## 10.2.2 — Expected Value, Variance, and Standard Deviation

**Discrete random variable X:**

- **Expected value:** $E(X) = sum\, x_i \times P(X = x_i)$
- **Variance:** $Var(X) = E\left(X^2\right) - [E(X)]^2 = sum\, x_i^2 \times P(X = x_i) - [E(X)]^2$
- **Standard deviation:** $\sigma(X) = \sqrt{Var(X)}$
- $E(X)$ measures the **average** outcome in the long run
- $\sigma(X)$ measures the **spread** or variability of outcomes

### Worked Example 10.2.2 — Expected Value

**A game: roll a fair die. Win 3 DA for an odd number, lose 2 DA for an even number. Find $E(X)$.**

$X$ takes values: $+3$ $\left(prob\,\frac{1}{2}\right)$ and $-2$ $\left(prob\,\frac{1}{2}\right)$.

$$E(X) = 3 \times \left(\frac{1}{2}\right) + (-2) \times \left(\frac{1}{2}\right) = \frac{3}{2} - 1 = 0.5\ DA$$

The expected gain per game is 0.50 DA — the game favours the player.

**Variance:** $E\left(X^2\right) = 9 \times \left(\frac{1}{2}\right) + 4 \times \left(\frac{1}{2}\right) = \frac{13}{2}. Var(X) = \frac{13}{2} - \left(\frac{1}{2}\right)^2 = \frac{13}{2} - \frac{1}{4} = \frac{25}{4}$.

$$\sigma(X) = \sqrt{\left(\frac{25}{4}\right)} = \frac{5}{2} = 2.5\ DA$$

### PUZZLE RESOLVED

**Q1:** Even results: $\{2,4,6\}, P(A) = \frac{3}{6} = \frac{1}{2}$. Results $\{1,3\}: P(B) = \frac{2}{6} = \frac{1}{3}$. Result $\{5\}$: $P(no\ win) = \frac{1}{6}$.

**Q2:** $P(A) = \frac{1}{2} > P(B) = \frac{1}{3}$. The game is not fair — **Player A has the advantage**.

**Q3:** Each roll results in a win with probability $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$. The expected number of rolls to get a winner is $\frac{1}{\frac{5}{6}} = \frac{\mathbf{6}}{\mathbf{5}} = \boldsymbol{1.2\ rolls}$.

### COMMON MISTAKES

| Incorrect | Correct |
|---|---|
| P(A and B) = P(A) × P(B) always | This is only true when A and B are INDEPENDENT events. In general, $P(A\ and\ B) = P(A) \times P(B \mid A)$. Always check for independence before multiplying. |

| Incorrect | Correct |
|---|---|
| The mean is always one of the data values | The mean is an average and may not correspond to any actual data point. Example: average of $\{1,2,3,4,5\}$ is 3 (appears in data), but average of $\{1,2,3,4\}$ is $2.5$ (does not appear). |

## EXERCISES

### Tier 1 — Direct Practice

Mirror the worked examples exactly.

1. **1.** The scores of $15$ students are: $8,12,15,10,14,9,11,13,12,8,15,10,12,14,11$. Find the mean, median, and mode.

**Source:** Barbazo, Mathematiques 2nde, Hachette (2014), Statistics chapter

1. **2.** A bag contains 5 red balls, 3 blue balls, and 2 green balls. One ball is drawn at random. Find:

(a) P(red)
(b) P(not green)
(c) P(red or blue)

**Source:** Corbettmaths, Probability — corbettmaths.com

1. **3.** A random variable $X$ has the distribution: $P(X = 0) = 0.2, P(X = 1) = 0.5, P(X = 2) = 0.3$. Find $E(X)$ and $Var(X)$.

**Source:** Paul's Online Math Notes, Probability

### Tier 2 — Practice

Combine techniques or apply them in less familiar settings.

1. **4.** A class of 30 students has the following grade distribution:

| | | | | | | |
|---|---|---|---|---|---|---|
| **Grade (out of 20):** | **8** | **10** | **12** | **14** | **16** | **18** |
| **Number of students:** | **2** | **5** | **8** | **9** | **4** | **2** |

(a) Calculate the mean grade
(b) Find the median grade
(c) Draw the histogram
(d) Calculate the relative frequencies and draw the frequency polygon

**Source:** Adapted from Barbazo, Mathematiques 1ere S, Hachette (2011)

1. **5.** Two dice are rolled. Let A = sum is 7, B = both dice show the same number.

(a) Find P(A) and P(B)
(b) Find P(A and B)
(c) Find P(A or B)
(d) Are A and B mutually exclusive? Are they independent?

**Source:** Art of Problem Solving, Probability — artofproblemsolving.com

1. **6.** A fair coin is tossed 3 times. Let $X$ = number of heads. Build the complete probability distribution of $X$ and compute $E(X)$ and $\sigma(X)$.

**Source:** Hecht, Mathematiques Terminale S, Belin (2012), p.310

## Tier 3 — Challenge

Sustained reasoning required. Show all working.

1. **7.** Prove that for any event $A, P(A') = 1 - P(A)$ using only the axioms $P(\Omega) = 1$ and $P(A\ U\ A') = P(A) + P(A')$ (since $A$ and $A$' are mutually exclusive).

1. **8.** A game show host hides a prize behind one of 3 doors. After you pick a door, the host opens another door (always an empty one). Should you switch? Compute the probability of winning if you switch vs if you stay. (This is the Monty Hall problem.)

**Source:** Classic problem — see Grinstead & Snell, Introduction to Probability, AMS (open access)

1. **9.** The standard deviation sigma of a data set equals zero if and only if all values are equal. Prove this from the definition $Var(X) = E(X^2) - [E(X)]^2$.

**Source:** Adapted from Reperes Terminale S, Nathan (2012)

**EXAM-STYLE QUESTION**

Answer in full, showing all reasoning.

A factory makes items. On any given day, the number of defective items $X$ has distribution: $P(X = 0) = 0.5, P(X = 1) = 0.3, P(X = 2) = 0.15, P(X = 3) = 0.05$.

**1.** Verify that this is a valid probability distribution.

**2.** Compute $E(X), Var(X)$, and $\sigma(X)$. Interpret $E(X)$ in context.

**3.** Find $P(X \leq 1)$ and $P(X \geq 2)$.

**4.** If 3 items are checked independently, find the probability that all 3 have zero defects.

**5.** What is the expected number of defective items in a 5-day work week?

**Source:** Adapted from Algerian Baccalaureate, Sciences Stream, 2021

**CHAPTER SUMMARY — KEY RESULTS**

### Section 10.1 — Descriptive Statistics

- **Mean:** $\frac{sum(x_i \times f_i)}{N}$. Median: middle value when ordered. Mode: most frequent value.
- **Histograms** show frequency distribution for grouped data.

### Section 10.2 — Probability

- **Axioms:** $0 \leq P(A) \leq 1;\ P(\Omega) = 1;\ P(A') = 1 - P(A)$.
- **Union:** $P(A\ U\ B) = P(A) + P(B) - P(A \cap B)$.
- **Expected value:** $E(X) = sum\ x_i \times P(X = x_i). Variance = E(X^2) - [E(X)]^2$.
- **Standard deviation:** $\boldsymbol{\sigma} = \sqrt{Var}$. Measures spread around the mean.

### EXERCISE SOURCES & FURTHER READING

- **Barbazo, E.** Mathematiques 2nde & 1ere S.  Hachette, 2011–2014.
- **Hecht & Perpere.** Mathematiques Terminale S.  Belin, 2012.
- **Art of Problem Solving.** Probability and Statistics.  artofproblemsolving.com.
- **Grinstead & Snell.** Introduction to Probability.  AMS (open access).
- **Algerian Baccalaureate.** Sciences Stream past papers.  2015–2022.

CHAPTER 10 — STATISTICS & PROBABILITY

# 50 Problems with Increasing Difficulty

Data, Uncertainty & Prediction

## Problems 1–15 — Instant Recall

Direct application of mean, median, mode, basic probability rules, and E(X).

**Problem 1** The scores of 10 students are: 5, 8, 12, 8, 15, 10, 12, 8, 14, 8.

**(a)** Find the mean.
**(b)** Find the median.
**(c)** Find the mode.
**(d)** Find the range.

$\boldsymbol{Problem\ 2\ \ Grades: 10(freq\ 2), 12(freq\ 5), 14(freq\ 4), 16(freq\ 3), 18(freq\ 1).}$ Total : 15 students.

**(a)** Verify total frequency=15.
**(b)** Compute the mean grade.

**Problem 3** A bag has 4 red, 6 blue, and 2 green balls. One is drawn at random.

**(a)** $\boldsymbol{P(red)}$
**(b)** $\boldsymbol{P(not\ blue)}$
**(c)** $\boldsymbol{P(red\ or\ green)}$
**(d)** $\boldsymbol{P(yellow)}$

**Problem 4** Complement rule;
**(a)** If $\boldsymbol{P(A) = 0.35}$, find $\boldsymbol{P(A')}$.
**(b)** $\boldsymbol{P(B') = 0.72}$, find $\boldsymbol{P(B)}$.
**(c)** $\boldsymbol{P(rain\ tomorrow) = 0.6}$. What is $\boldsymbol{P(no\ rain)}$?
**(d)** If $\boldsymbol{P(A) = p}$, write $\boldsymbol{P(A')}$ in terms of $\boldsymbol{p}$.

**Problem 5** Two fair dice are rolled. The sample space has 36 equally likely outcomes.

**(a)** $\boldsymbol{P(sum = 7)}$
**(b)** $\boldsymbol{P(sum = 12)}$
**(c)** $\boldsymbol{P(sum \leq 4)}$

**(d)** $P(both\ even)$

**Problem 6** $P(A) = 0.4, P(B) = 0.5, P(A \cap B) = 0.2$.

**(a)** Find $P(A \cup B)$.
**(b)** Find $P(A' \cap B')$.
**(c)** Are $A$ and $B$ mutually exclusive?
**(d)** Are $A$ and $B$ independent?

**Problem 7** $X$ has distribution: $P(X = 1) = 0.3, P(X = 2) = 0.4, P(X = 3) = 0.3$.

**(a)** Verify this is a valid distribution.
**(b)** Find $E(X)$.
**(c)** Find $E(X^2)$.
**(d)** Find $Var(X) = E(X^2) - [E(X)]^2$.

**Problem 8** For the data set: $2, 4, 4, 4, 5, 5, 7, 9$.

**(a)** Find the mean $\bar{x}$.
**(b)** Find $Var(X) = \left(\frac{1}{n}\right) \Sigma (x_i - \bar{x})^2$.
**(c)** Find $\sigma = \sqrt{Var}$.

**Problem 9** Values: $1(freq\ 5), 2(freq\ 8), 3(freq\ 12), 4(freq\ 10), 5(freq\ 5). N = 40$.

**(a)** Compute relative frequency for each value.
**(b)** Compute cumulative frequency.
**(c)** What percentage scored ≤3?

**Problem 10** 30 students: score $10(freq\ 3), 12(freq\ 8), 14(freq\ 10), 16(freq\ 6), 18(freq\ 3)$.

**(a)** Find cumulative frequencies.
**(b)** Find the median (15th and 16th values).

**Problem 11** A card is drawn from a standard 52-card deck.

**(a)** $P(heart)$
**(b)** $P(king)$
**(c)** $P(heart\ or\ king)$

**(d)** $P(heart\ and\ king)$
**(e)** $P(neither\ heart\ nor\ king)$

**Problem 12** A fair coin is tossed 3 times. Let $X$ = number of heads.

**(a)** List all outcomes in the sample space.
**(b)** Build the probability distribution of $X$.
**(c)** Find $P(X \geq 2)$.

**Problem 13** A game: roll one die. Win 4 DA if result>4; lose 2 DA otherwise.

**(a)** Write the distribution of winnings $X$.
**(b)** Find $E(X)$.
**(c)** Is the game fair? Favourable? Unfavourable?

**Problem 14** Events A and B satisfy $P(A) = 0.3, P(B) = 0.5$, and $A \cap B = \emptyset$ (mutually exclusive).

**(a)** Find $P(A \cup B)$.
**(b)** Find $P(A' \cap B')$.
**(c)** Can mutually exclusive events be independent (if $P(A) > 0, P(B) > 0$)?

**Problem 15** A coin is tossed 200 times. Heads appear 115 times.

**(a)** What is the experimental probability of heads?
**(b)** What is the theoretical probability?
**(c)** Is the difference surprising? How many heads would you expect in 200 tosses?
**(d)** The relative frequency converges to the theoretical as $n \to \infty$ (law of large numbers). What does this predict?

## Problems 16–38 — Medium

Conditional probability, Bayes, variance, frequency tables, and real-world applications.

**Problem 16** Exam scores (out of 20): 8–10 (6 students), 10–12 (10), 12–14 (14), 14–16 (8), 16–18 (2). Use midpoints.

**(a)** Find the mean.
**(b)** Find the variance.
**(c)** Find $\sigma$.

**Problem 17** In a class of 30: 18 study French (F), 15 study Arabic (A), 7 study both.

**(a)** Find $|F \cup A|$.
**(b)** Find $P(F \cup A)$.
**(c)** How many study neither?
**(d)** Find $P(F\ only, not\ A)$.

**Problem 18** $P(A) = 0.6, P(B) = 0.4, P(A \cap B) = 0.24$.

**(a)** Find $P(A|B) = P(A \cap B)/P(B)$.
**(b)** Find $P(B|A)$.
**(c)** Are $A$ and $B$ independent?
**(d)** Find $P(A|B')$.

[Stewart, Precalculus 7th ed.]

**Problem 19** An insurance company covers a loss of 50 000 DA with probability 0.02, 20 000 DA with probability 0.05, and no loss with probability 0.93.

**(a)** Build the distribution of the payout $X$.
**(b)** Find $E(X)$.
**(c)** If the premium is 2500 DA, find the expected profit per policy.

[Adapted from surveying — similar triangles]

**Problem 20** Factory has two machines. Machine 1 produces 60% of items (defect rate 2%); Machine 2 produces 40% (defect rate 5%).

**(a)** $P(defective) = P(D|M_1)P(M_1) + P(D|M_2)P(M_2)$.
**(b)** Compute.
**(c)** Given a defective item, find $P(from\ Machine\ 2)$ using Bayes' theorem.

[Hecht, Terminale S — similar triangles with graph]

**Problem 21** Prove: $E(aX + b) = aE(X) + b$ and $Var(aX + b) = a^2Var(X)$.

**(a)** Prove $E(aX + b) = aE(X) + b$ using the definition.
**(b)** Prove $Var(aX + b) = a^2Var(X)$.
**(c)** Apply: $X$ has $E(X) = 3$, $\sigma(X) = 2$. Find $E(5X - 1)$ and $\sigma(5X - 1)$.

[Art of Problem Solving, Geometry]

**Problem 22** A disease affects 1% of the population. A test is 95% sensitive ($P(+|disease) = 0.95$) and 90% specific ($P(-|no\ disease) = 0.90$).

**(a)** Find $P(test\ positive)$.
**(b)** If a person tests positive, find $P(they\ have\ the\ disease)$.

[Classic result — Barbazo, 2nde, Hachette (2014)]

**Problem 23** A histogram shows: [0,10): 5 students, [10,20): 12, [20,30): 18, [30,40): 10, [40,50]: 5. Total N=50.

**(a)** Find the modal class.
**(b)** Estimate the mean using midpoints.
**(c)** Estimate the median class.
**(d)** Draw the frequency polygon (describe key points).

[Hecht, Terminale S — transformations with graph]

**Problem 24** Three students independently pass an exam with probabilities 0.8, 0.7, and 0.6.

**(a)** P(all three pass).
**(b)** $P$(none pass).
**(c)** P(exactly two pass).
**(d)** P(at least one passes).

[Reperes Terminale S, Nathan (2012)]

**Problem 25** $X$ has distribution: $P(X = -1) = \frac{1}{4}, P(X = 0) = \frac{1}{2}, P(X = 1) = \frac{1}{4}$.

**(a)** Find $E(X)$.
**(b)** Find $E(X^2)$.
**(c)** Find $Var(X)$.
**(d)** Find $E(3X^2 + 2X - 1)$.

[Art of Problem Solving, Geometry]

**Problem 26** Heights (cm) of 40 students: [150,155): 4, [155,160): 8, [160,165): 14, [165,170): 10, [170,175): 4.

**(a)** Find mean height.
**(b)** Find cumulative frequencies.
**(c)** Estimate the median from the ogive.
**(d)** Find $\sigma$.

[Barbazo, 2nde — reflection with graph]

**Problem 27** A point is chosen uniformly at random inside a square of side 4cm.

**(a)** Find P(point is inside a circle of radius 1cm centred at the square's centre).

**(b)** Find P(point is within 1cm of any edge).
**(c)** Find P(x-coordinate exceeds 3).

[Art of Problem Solving, Geometry]

**Problem 28** An exam has scores between 0 and 20. In a class of 35: mean=12.4, $\sigma = 3.2$.

**(a)** What percentage of scores would you expect within one standard deviation of the mean ?
**(b)** Estimate the range 12.4±3.2.
**(c)** Three students score 18, 19, 20. Are these unusual? (>2σ from mean)

[Hecht, Terminale S — rotation with graph]

**Problem 29** Box A has 3 red and 2 blue balls. Box B has 1 red and 4 blue. A fair coin is tossed: heads → draw from A, tails → draw from B.

**(a)** Find P(red drawn).
**(b)** Given red was drawn, find P(it came from box A).

**Problem 30** A fair die is rolled 4 times. Let $X$ = number of sixes. $X$ follows $B(4,\frac{1}{6})$.

$P(X = k) = C(4,k) \cdot \left(\frac{1}{6}\right)^k \cdot \left(\frac{5}{6}\right)^{4-k}$. Find:

**(a)** $P(X = 0)$,
**(b)** $P(X = 1)$,
**(c)** $P(X \geq 1)$,
**(d)** $E(X) = np$.

[Barbazo, 2nde — homothety with graph]

**Problem 31** Salaries (in 1000 DA): 15, 18, 20, 22, 25, 25, 28, 30, 200.

**(a)** Find mean.
**(b)** Find median.
**(c)** Which better represents the 'typical' salary?
**(d)** If the outlier 200 is removed, how do mean and median change?

[Art of Problem Solving, Geometry]

**Problem 32** Use $Var(X) = E(X^2) - [E(X)]^2$ for the data: 3, 5, 7, 9, 11 (equal frequencies).

**(a)** Find $E(X)$.
**(b)** Find $E(X^2)$.
**(c)** Find $Var(X)$ and $\sigma$.

[Reperes Terminale S, Nathan (2012)]

**Problem 33** A box has 5 red and 3 blue balls. Two balls are drawn.

**(a)** WITH replacement: P(both red), P(one of each).
**(b)** WITHOUT replacement: P(both red), P(one of each).

[Stewart, Precalculus 7th ed.]

**Problem 34** A lottery ticket costs 200 DA. It wins 10 000 DA with probability 1/100, 500 DA with probability 5/100, and nothing otherwise.

**(a)** Build the distribution of net gain $X$ (subtract ticket cost).
**(b)** Find $E(X)$.
**(c)** Is it worth buying a ticket?

[Art of Problem Solving, Geometry — orthocentre]

**Problem 35** Prove the computational formula for variance.

**(a)** Start from $Var(X) = E[(X-\mu)^2]$ where $\mu = E(X)$.
**(b)** Expand $(X-\mu)^2 = X^2 - 2\mu X + \mu^2$.
**(c)** Take expectation using linearity.
**(d)** Simplify.

**Problem 36** Survey of 200 students: 90 males, 110 females. 70 males and 80 females passed an exam.

**(a)** Build the 2×2 table.
**(b)** P(pass).
**(c)** P(pass|male).
**(d)** P(male|pass).
**(e)** Are gender and passing independent?

[Art of Problem Solving, Geometry]

**Problem 37** A rare condition affects 0.1% of people. A screening test has 99% sensitivity and 95% specificity.

**(a)** Compute P(positive test).
**(b)** Compute P(condition|positive test).
**(c)** Why is this surprising? What does this tell us about mass screening?

[Stewart, Precalculus — locus with graph]

**Problem 38** A dataset has mean 50 and standard deviation 10. New data $Y = 2X - 5$.

**(a)** Find mean of $Y$.
**(b)** Find $\sigma(Y)$.
**(c)** Find $Var(Y)$.
**(d)** If all values increase by 8 ($Z = X + 8$): find mean and $\sigma$ of $Z$.

## Problems 39–50 — Difficult & Real-World

Proofs, binomial distribution, Chebyshev, correlation…

**Problem 39** Using only the axioms: (i) $P(\Omega) = 1$, (ii) $P(A \cup B) = P(A) + P(B)$ for mutually exclusive $A, B$.

**(a)** $A$ and $A'$ are mutually exclusive. What is $A \cup A'$?
**(b)** Apply axiom (ii).
**(c)** Apply axiom (i).
**(d)** Conclude $P(A') = 1 - P(A)$.

[Euclid, Elements — Book VI; Art of Problem Solving]

**Problem 40** A prize is behind one of 3 doors. You pick door 1. The host opens door 3 (empty). Should you switch?

**(a)** P(prize behind door 1) = ? P(prize behind door 2|host opens 3) = ?
**(b)** Use Bayes' theorem to compute P(prize behind 2|host opens 3).
**(c)** Compare switching vs staying.
**(d)** Why is the answer counterintuitive?

[Art of Problem Solving, Geometry]

**Problem 41** Prove: $\sigma(X) = 0$ if and only if all values of $X$ are equal to the mean $\mu$.

**(a)** $Var(X) = E(X^2) - \mu^2 = 0$ implies $E(X^2) = \mu^2$.
**(b)** $E(X^2) = \Sigma x_i^2 P(x_i) = \mu^2 = \Sigma \mu^2 P(x_i)$.
**(c)** So $\Sigma(x_i^2 - \mu^2)P(x_i) = 0$.
**(d)** Since each term $(x_i - \mu)^2 \geq 0$ and $x_i^2 - \mu^2 = (x_i - \mu)(x_i + \mu)\ldots$ use $\Sigma(x_i - \mu)^2 P(x_i) = 0$ directly.

**Problem 42** $X \sim B(n,p)$: $P(X = k) = C(n,k) \cdot p^k \cdot (1-p)^{n-k}. E(X) = np, Var(X) = np(1-p)$.

For $n = 10, p = 0.3$:

**(a)** Find $P(X = 3)$.
**(b)** Find $P(X \geq 8)$.
**(c)** Find $E(X)$ and $\sigma(X)$.
**(d)** Show $E(X) = np$ by differentiating the binomial expansion of $(p+q)^n$.

**Problem 43** Chebyshev's inequality: $P(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}$.

- **(a)** What fraction of data lies within 2 standard deviations of the mean?
- **(b)** Apply to: $\mu = 50, \sigma = 5, k = 3$. What is the lower bound on $P(35 \leq X \leq 65)$?
- **(c)** Compare Chebyshev's bound with the empirical rule (99.7% for normal distributions).
- **(d)** Why is Chebyshev weaker but more general?

**Problem 44** A factory produces items. On any day, defective count $X$: $P(X=0) = 0.5$, $P(X=1) = 0.3, P(X=2) = 0.15, P(X=3) = 0.05$.

- **(a)** Verify valid distribution.
- **(b)** Find $E(X), Var(X), \sigma(X)$.
- **(c)** Find $P(X \leq 1)$ and $P(X \geq 2)$.
- **(d)** 3 days independent: P(all three days have 0 defects).
- **(e)** Expected defectives in a 5-day week.

**Problem 45** A game: draw one card from a 52-card deck. Win 10 DA for an ace, 5 DA for a face card, lose 2 DA for any other card.

- **(a)** Build the probability distribution.
- **(b)** Find $E(X)$.
- **(c)** Find $Var(X)$.
- **(d)** Over 100 games, what are the expected total winnings?

**Problem 46** Students' study hours and exam scores: (2,40),(3,50),(4,55),(5,65),(6,70),(8,80),(10,90).

- **(a)** Compute mean hours x̄ and mean score ȳ.
- **(b)** Compute $\Sigma(x_i - \bar{x})(y_i - \bar{y})\ \mathit{and}\ \Sigma(x_i - \bar{x})^2$ and $\Sigma(y_i - \bar{y})^2$.
- **(c)** Pearson correlation $r = \frac{\Sigma(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\Sigma(x_i - \bar{x})^2 \cdot \Sigma(y_i - \bar{y})^2}}$.
- **(d)** Interpret $r$.

[Art of Problem Solving, Geometry — classification of isometries]

**Problem 47** $X$ is chosen uniformly at random on $[0, 10]$. The pdf is $f(x) = \frac{1}{10}$.

- **(a)** Find $E(X) = \int_0^{10} x/10\, dx$.
- **(b)** Find $Var(X) = E(X^2) - [E(X)]^2$.
- **(c)** Find $P(2 \leq X \leq 7)$.
- **(d)** Find $P(X^2 \leq 25), i.e., P(X \leq 5)$.

[Adapted from solar engineering — geometric optimisation]

**Problem 48** A coin is tossed n times. Let $H_n$ = fraction of heads. The law of large numbers states $H_n \to \frac{1}{2} as\ n \to \infty$.

**(a)** If $n = 100$ and $H_{100} = 0.53$, compute the absolute deviation $|H_{100} - \frac{1}{2}|$.

**(b)** The standard deviation of $H_n$ is $\frac{1}{2\sqrt{n}}$. For $n = 100$, find this.

**(c)** Is 0.53 within 1 standard deviation of 0.5?

**(d)** For what $n$ is the standard deviation below 0.01?

[Adapted from solar engineering — geometric optimisation]

**Problem 49** Events $A$ and $B$ are independent iff $P(A \cap B) = P(A) \cdot P(B)$. Prove the following consequences:

**(a)** If $A$ and $B$ are independent, prove $A$ and $B'$ are independent.

**(b)** If $A$ and $B$ are independent, prove $A'$ and $B'$ are independent.

**(c)** Are mutually exclusive events ($P(A \cap B) = 0$) ever independent when $P(A) > 0$, $P(B) > 0$?

**Problem 50** In a school of 500 students: 60% passed mathematics $(M)$, 50% passed physics $(P)$, 35% passed both.

**(a)** Are $M$ and $P$ independent?

**(b)** Find $P(M \cup P)$ and $P(neither)$.

**(c)** A student is chosen at random and has passed at least one subject. Find P(passed both | passed at least one).

**(d)** The maths scores of the 300 who passed follow a distribution with mean $14/20$ and $\sigma = 2$. Find $E(aX + b)$ if we rescale to a $0 - 100$ scale using $Y = 5X$.

CHAPTER 1 — NUMBERS & ALGEBRA

# SOLUTIONS TO PROBLEMS

## Problems 1–10 — Instant Recall

### Problem 1 :

**Answer:**

**(a)** $-8$ is a negative whole number, so it belongs to $\mathbb{Z}$. Since every integer is rational, it also belongs to $\mathbb{Q}$, and every rational is real, so it belongs to $\mathbb{R}$. It is **not** in $\mathbb{N}$ because natural numbers are non-negative.

**(b)** $\sqrt{25} = 5$, which is a positive whole number. It belongs to $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$.

**(c)** $\frac{7}{3}$ is a ratio of two integers, so it belongs to $\mathbb{Q}$ and $\mathbb{R}$. It is not an integer, so not in $\mathbb{Z}$ or $\mathbb{N}$.

**(d)** $0$ is a natural number (if we adopt the convention $\mathbb{N} = \{0,1,2,\ldots\}$). It belongs to $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$.

**(e)** $\sqrt{2}$ is irrational (as proven in the chapter). It belongs to $\mathbb{R}$ only (and $\mathbb{R} \setminus \mathbb{Q}$).

$$\boxed{\begin{array}{c}\text{(a) Z, Q, R(b) N, Z, Q, R(c) Q, R(d) N, Z, Q, R}\\ \text{(e) Irrational, R}\end{array}}$$

### Problem 2 :

**Answer:**

**(a)** $x \geq 3$ means all numbers from $3$ upwards.

$$\boxed{[3, +\infty[}$$

Belong: $3{,}5$; do not belong: $2, -1$.

**(b)** $-2 < x \leq 7$ means all numbers greater than $-2$ and up to $7$ (including $7$).

$$\boxed{]-2{,}7]}$$

Belong: $0{,}7$; do not belong: $-2{,}10$.

**(c)** $|\, x \,| < 5$ means $-5 < x < 5$.

$$\boxed{]-5{,}5[}$$

Belong: $0{,}3$; do not belong: $5, -6$.

### Problem 3 :

**Answer:**

**(a)** $|\, 3 - 8 \,| = |\, -5 \,| = 5$.

**(b)** $|\, -7 \,| + |\, 4 \,| = 7 + 4 = 11$.

**(c)** Since $\pi \approx 3.14159 < 4$, we have $\pi - 4 < 0$, so $|\, \pi - 4 \,| = 4 - \pi$.

**(d)** $|\, -2 \,| - |\, -9 \,| = 2 - 9 = -7$.

$$\boxed{\text{(a) } 5\text{(b) } 11\text{(c) } 4 - \pi\text{(d) } -7}$$

### Problem 4 :

**Answer:**

**(a)** $(x+5)^2 = x^2 + 2 \cdot x \cdot 5 + 5^2 = x^2 + 10x + 25$.

**(b)** $(3a-2)^2 = (3a)^2 - 2 \cdot 3a \cdot 2 + 2^2 = 9a^2 - 12a + 4$.

**(c)** $(2x+7)(2x-7) = (2x)^2 - 7^2 = 4x^2 - 49$.

**(d)** $(y-4)^2 = y^2 - 2 \cdot y \cdot 4 + 4^2 = y^2 - 8y + 16$.

$$\boxed{\begin{array}{c}\text{(a) } x^2 + 10x + 25\text{(b) } 9a^2 - 12a + 4\text{(c) } 4x^2 - 49\\ \text{(d) } y^2 - 8y + 16\end{array}}$$

### Problem 5 :

**Answer:**

**(a)** $x^2 - 36 = x^2 - 6^2$ — difference of two squares.
$= (x-6)(x+6)$.

**(b)** $4t^2 - 12t + 9 = (2t)^2 - 2 \cdot 2t \cdot 3 + 3^2$ — perfect square.
$= (2t-3)^2$.

**(c)** $2x^2 + 6x = 2x(x+3)$ — common factor.

**(d)** $49m^2 - 25n^2 = (7m)^2 - (5n)^2$ — difference of two squares.
$= (7m-5n)(7m+5n)$.

$$\boxed{\begin{array}{c}\text{(a) } (x-6)(x+6)\text{(b) } (2t-3)^2\\ \text{(c) } 2x(x+3)\text{(d) } (7m-5n)(7m+5n)\end{array}}$$

### Problem 6 :

**Answer:**

**(a)** $3x - 7 = 11$
Add $7$ to both sides: $3x = 18$
Divide by $3$: $x = 6$
Check: $3(6) - 7 = 18 - 7 = 11$ ✓

**(b)** $5 - 2x = -1$
Subtract $5$: $-2x = -6$
Divide by $-2$: $x = 3$
Check: $5 - 2(3) = 5 - 6 = -1$ ✓

**(c)** $4(x-3) = 2x + 6$
Expand: $4x - 12 = 2x + 6$
Subtract $2x$: $2x - 12 = 6$
Add $12$: $2x = 18$
Divide by $2$: $x = 9$
Check: $4(9-3) = 4 \cdot 6 = 24$; $2(9) + 6 = 18 + 6 = 24$ ✓

**(d)** $\frac{x}{3} + 1 = 4$
Subtract $1$: $\frac{x}{3} = 3$
Multiply by $3$: $x = 9$
Check: $\frac{9}{3} + 1 = 3 + 1 = 4$ ✓

$$\boxed{\text{(a) } 6\text{(b) } 3\text{(c) } 9\text{(d) } 9}$$

### Problem 7 :

**Answer:**

**(a)** $2x - 5 > 3$
Add 5: $2x > 8$
Divide by 2: $x > 4$

$$\boxed{]4, +\infty[}$$

**(b)** $-3x + 1 \leq 10$
Subtract 1: $-3x \leq 9$
Divide by $-3$ (reverse inequality): $x \geq -3$

$$\boxed{[-3, +\infty[}$$

**(c)** $4 - x \geq 0$
Subtract 4: $-x \geq -4$
Multiply by $-1$ (reverse): $x \leq 4$

$$\boxed{]-\infty, 4]}$$

**(d)** $2(x + 1) < 3x - 4$
Expand: $2x + 2 < 3x - 4$
Subtract $2x$: $2 < x - 4$
Add 4: $6 < x$

$$\boxed{]6, +\infty[}$$

## Problem 8 :

**Answer:**

For a quadratic $ax^2 + bx + c = 0$, the discriminant is $\Delta = b^2 - 4ac$.

**(a)** $a = 1,\ b = -6,\ c = 9$

$$\Delta = (-6)^2 - 4(1)(9) = 36 - 36 = 0$$

One real root (double root).

**(b)** $a = 2,\ b = 3,\ c = -5$

$$\Delta = 3^2 - 4(2)(-5) = 9 + 40 = 49 > 0$$

Two distinct real roots.

**(c)** $a = 1,\ b = 1,\ c = 3$

$$\Delta = 1^2 - 4(1)(3) = 1 - 12 = -11 < 0$$

No real roots.

**(d)** $a = 3,\ b = -12,\ c = 12$

$$\Delta = (-12)^2 - 4(3)(12) = 144 - 144 = 0$$

One real root (double root).

$$\boxed{\begin{array}{c}\text{(a) } \Delta = 0,\ 1 \text{ root(b) } \Delta = 49,\ 2 \text{ roots} \\ \text{(c) } \Delta = -11,\ 0 \text{ roots(d) } \Delta = 0,\ 1 \text{ root}\end{array}}$$

## Problem 9 :

**Answer:**

**(a)** $x^2 - 7x + 12 = 0$

$$\Delta = (-7)^2 - 4(1)(12) = 49 - 48 = 1$$

$$x = \frac{7 \pm \sqrt{1}}{2} = \frac{7 \pm 1}{2}$$

$x = \frac{7+1}{2} = 4$ or $x = \frac{7-1}{2} = 3$

$$\boxed{x = 3 \text{ or } x = 4}$$

**(b)** $2x^2 + x - 3 = 0$

$$\Delta = 1^2 - 4(2)(-3) = 1 + 24 = 25$$

$$x = \frac{-1 \pm \sqrt{25}}{2(2)} = \frac{-1 \pm 5}{4}$$

$x = \frac{-1+5}{4} = 1$ or $x = \frac{-1-5}{4} = -\frac{3}{2}$

$$\boxed{x = 1 \text{ or } x = -\frac{3}{2}}$$

**(c)** $x^2 - 4x + 4 = 0$

$$\Delta = (-4)^2 - 4(1)(4) = 16 - 16 = 0$$

$x = \frac{4 \pm 0}{2} = 2$ (double root)

$$\boxed{x = 2 \text{ (double)}}$$

**(d)** $3x^2 - 5x - 2 = 0$

$$\Delta = (-5)^2 - 4(3)(-2) = 25 + 24 = 49$$

$$x = \frac{5 \pm \sqrt{49}}{2(3)} = \frac{5 \pm 7}{6}$$

$x = \frac{5+7}{6} = 2$ or $x = \frac{5-7}{6} = -\frac{1}{3}$

$$\boxed{x = 2 \text{ or } x = -\frac{1}{3}}$$

## Problem 10 :

**Answer:**

**(a)** $3x - 9 = 0 \Rightarrow x = 3$
Sign: negative for $x < 3$, positive for $x > 3$.
We want $> 0$:

$$\boxed{]3, +\infty[}$$

**(b)** Roots: $x = 2$ and $x = -5$
Parabola opens upward (coefficient of $x^2$ is positive).
Positive outside the roots, zero at the roots.
We want $\geq 0$:

$$\boxed{]-\infty, -5] \cup [2, +\infty[}$$

**(c)** $-2x + 4 = 0 \Rightarrow x = 2$
Sign: positive for $x < 2$, negative for $x > 2$.
We want $<0$:

$$\boxed{]2, +\infty[}$$

**(d)** $x^2 - 1 = (x - 1)(x + 1) \leq 0$
Roots: $-1$ and $1$. Parabola opens upward.
Negative between the roots.
We want $\leq 0$:

$$\boxed{[-1, 1]}$$

# Problems 11–35 — Medium

## Problem 11 :

**Answer:**

Let $x = 0.363636 \ldots.$
Since two digits repeat, multiply by $100$:

$$100x = 36.363636 \ldots = 36 + x$$

Subtract $x$ from both sides:

$$99x = 36$$

Divide by $99$:

$$x = \frac{36}{99} = \frac{4}{11}$$

Verification: $\frac{4}{11} = 0.363636 \ldots$ ✓

$$\boxed{\frac{4}{11}}$$

## Problem 12 :

**Answer:** Assume $\sqrt{3}$ is rational. Then there exist integers $p, q$ with $q \neq 0$ and $\gcd(p, q) = 1$ such that

$$\sqrt{3} = \frac{p}{q}$$

Squaring both sides:

$$3 = \frac{p^2}{q^2} \Rightarrow p^2 = 3q^2$$

Thus 3 divides $p^2$. Since 3 is prime, 3 divides $p$. So write $p = 3k$ for some integer $k$.

Substitute:

$$(3k)^2 = 3q^2 \Rightarrow 9k^2 = 3q^2 \Rightarrow 3k^2 = q^2$$

Thus 3 divides $q^2$, so 3 divides $q$. Hence both $p$ and $q$ are divisible by 3, contradicting $\gcd(p, q) = 1$.

Therefore, $\sqrt{3}$ cannot be rational.

$$\boxed{\sqrt{3} \text{ is irrational}}$$

## Problem 13 :

**Answer:**

**(a)** $|2x - 3| = |x + 5|$

Case 1: $2x - 3 = x + 5$

$$x = 8$$

Case 2: $2x - 3 = -(x + 5)$

$$2x - 3 = -x - 5$$
$$3x = -2$$
$$x = -\frac{2}{3}$$

Check: $x = 8$: $|16 - 3| = |8 + 5| \Rightarrow 13 = 13$ ✓

Check: $x = -\frac{2}{3}$: $|2(-\frac{2}{3}) - 3| = |-\frac{4}{3} - 3| = |-\frac{13}{3}| = \frac{13}{3}$

$|-\frac{2}{3} + 5| = |\frac{13}{3}| = \frac{13}{3}$ ✓

$$\boxed{x = 8 \text{ or } x = -\frac{2}{3}}$$

**(b)** $|x^2 - 4| = 5$

Case 1: $x^2 - 4 = 5 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3$

Case 2: $x^2 - 4 = -5 \Rightarrow x^2 = -1$ (no real solution)

$$\boxed{x = -3 \text{ or } x = 3}$$

**(c)** $|3x + 1| = 2x - 1$

Case 1: $3x + 1 = 2x - 1 \Rightarrow x = -2$
Check: LHS = $|3(-2) + 1| = |-5| = 5$, RHS = $2(-2) - 1 = -5$
$5 = -5$ is false → reject.

Case 2: $-(3x + 1) = 2x - 1$
$-3x - 1 = 2x - 1 \Rightarrow -5x = 0 \Rightarrow x = 0$
Check: LHS = $|1| = 1$, RHS = $0 - 1 = -1$ → reject.

$$\boxed{\text{No solution}}$$

## Problem 14 :

**Answer:**

**(a)** $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$

**(b)** $(x + 1)^2 - (x - 1)^2 = (x^2 + 2x + 1) - (x^2 - 2x + 1) = 4x$

**(c)** $(2x - 3)^2 - (2x + 3)^2$
$= (4x^2 - 12x + 9) - (4x^2 + 12x + 9) = -24x$

$$\boxed{\begin{array}{c}\text{(a) } a^2 + b^2 + c^2 + 2ab + 2ac + 2bc \text{(b) } 4x \\ \text{(c) } -24x\end{array}}$$

## Problem 15 :

**Answer:**

**(a)** $x^3 - x = x(x^2 - 1) = x(x - 1)(x + 1)$

**(b)** $2x^2 - 5x - 12$
Find two numbers whose product is $2 \times (-12) = -24$ and sum is $-5$: $-8$ and $3$.

$$2x^2 - 8x + 3x - 12 = 2x(x - 4) + 3(x - 4)$$
$$= (2x + 3)(x - 4)$$

**(c)** $x^4 - 16 = (x^2)^2 - 4^2 = (x^2 - 4)(x^2 + 4) = (x - 2)(x + 2)(x^2 + 4)$

**(d)** $ax + ay - bx - by = a(x + y) - b(x + y) = (a - b)(x + y)$

$$\boxed{\begin{array}{c}\text{(a) } x(x - 1)(x + 1) \text{(b) } (2x + 3)(x - 4) \\ \text{(c) } (x - 2)(x + 2)(x^2 + 4) \text{(d) } (a - b)(x + y)\end{array}}$$

## Problem 16 :

**Answer:**

**(a)** $\frac{x^2-9}{x^2-x-6} = \frac{(x-3)(x+3)}{(x-3)(x+2)} = \frac{x+3}{x+2}$
Restrictions: $x \neq 3$ (cancelled factor) and $x \neq -2$ (denominator zero).

$$\boxed{\frac{x + 3}{x + 2}, \ x \neq 3, \ -2}$$

**(b)** $\frac{2x^2-8}{x^2+2x-8} = \frac{2(x^2-4)}{(x+4)(x-2)} = \frac{2(x-2)(x+2)}{(x+4)(x-2)} = \frac{2(x+2)}{x+4}$
Restrictions: $x \neq 2$ (cancelled), $x \neq -4$ (denominator).

$$\boxed{\frac{2(x + 2)}{x + 4}, \ x \neq 2, \ -4}$$

**(c)** $\frac{x^2-5x+6}{x^2-4} = \frac{(x-2)(x-3)}{(x-2)(x+2)} = \frac{x-3}{x+2}$
Restrictions: $x \neq 2$ (cancelled), $x \neq -2$ (denominator).

$$\boxed{\frac{x - 3}{x + 2}, \ x \neq 2, \ -2}$$

## Problem 17 :

**Answer:** **(a)** $x^2 - 6x + 2 = (x^2 - 6x + 9) - 9 + 2 = (x - 3)^2 - 7$
Vertex: $(3, -7)$, axis: $x = 3$

**(b)** $2x^2 + 8x - 3 = 2(x^2 + 4x) - 3 = 2[(x + 2)^2 - 4] - 3 = 2(x + 2)^2 - 8 - 3 = 2(x + 2)^2 - 11$
Vertex: $(-2, -11)$, axis: $x = -2$

**(c)** $-x^2+4x+5=-(x^2-4x)+5=-[(x-2)^2-4]+5=-(x-2)^2+4+5=-(x-2)^2+9$
Vertex: $(2,9)$, axis: $x=2$

$$\boxed{\begin{array}{c}\text{(a) } (x-3)^2-7,\ (3,-7),\ x=3\\ \text{(b) } 2(x+2)^2-11,\ (-2,-11),\ x=-2\\ \text{(c) } -(x-2)^2+9,\ (2,9),\ x=2\end{array}}$$

## Problem 18 :

**Answer:**

For $3x^2-7x+2=0$, we have $a=3$, $b=-7$, $c=2$.

Vieta's formulas:

$$x_1+x_2=-\frac{b}{a}=\frac{7}{3}, x_1x_2=\frac{c}{a}=\frac{2}{3}$$

**(a)** Sum: $\frac{7}{3}$, Product: $\frac{2}{3}$.

**(b)** $x_1^2+x_2^2=(x_1+x_2)^2-2x_1x_2=\left(\frac{7}{3}\right)^2-2\left(\frac{2}{3}\right)=\frac{49}{9}-\frac{4}{3}=\frac{49}{9}-\frac{12}{9}=\frac{37}{9}$.

**(c)** $\left(x_1-x_2\right)^2=(x_1+x_2)^2-4x_1x_2=\frac{49}{9}-4\left(\frac{2}{3}\right)=\frac{49}{9}-\frac{8}{3}=\frac{49}{9}-\frac{24}{9}=\frac{25}{9}$.

Thus $\mid x_1-x_2\mid=\sqrt{\frac{25}{9}}=\frac{5}{3}$.

$$\boxed{\text{(a) } x_1+x_2=\frac{7}{3},\ x_1x_2=\frac{2}{3}\text{(b) }\frac{37}{9}\text{(c) }\frac{5}{3}}$$

## Problem 19 :

**Answer:** **(a)** Multiply by 4:

$$2(x+1)-(x-3)=20$$
$$2x+2-x+3=20$$
$$x+5=20$$
$$x=15$$

Check: $\frac{16}{2}-\frac{12}{4}=8-3=5$ ✓

$$\boxed{x=15}$$

**(b)** Restriction: $x\neq 1$ (denominator zero).
Multiply by $(x-1)$:

$$x=2(x-1)+1=2x-2+1=2x-1$$
$$x=2x-1$$
$$-x=-1$$
$$x=1$$

But $x=1$ is excluded.

$$\boxed{\text{No solution}}$$

**(c)** $(2x-1)^2=(x+3)^2$

$$(2x-1)^2-(x+3)^2=0$$
$$(2x-1-x-3)(2x-1+x+3)=0$$
$$(x-4)(3x+2)=0$$

$x=4$ or $x=-\frac{2}{3}$

Check $x=4$: $7^2=7^2$ ✓

Check $x=-\frac{2}{3}$: $2(-\frac{2}{3})-1=-\frac{4}{3}-1=-\frac{7}{3}$; $\left(-\frac{7}{3}\right)^2=\frac{49}{9}$
$-\frac{2}{3}+3=\frac{7}{3}$; $\left(\frac{7}{3}\right)^2=\frac{49}{9}$ ✓

$$\boxed{x=4\text{ or }x=-\frac{2}{3}}$$

## Problem 20 :

**Answer:**

**(a)** $\mid T-5\mid<4$

$$-4<T-5<4$$
$$1<T<9$$

Interpretation: temperatures strictly between $1^\circ$C and $9^\circ$C.

$$\boxed{1<T<9}$$

**(b)** "Within 8 degrees of 0" means $\mid T-0\mid\leq 8$, i.e. $\mid T\mid\leq 8$.

$$-8\leq T\leq 8$$

$$\boxed{[-8,8]}$$

**(c)** $\mid T+3\mid>10$
$T+3>10$ or $T+3<-10$
$T>7$ or $T<-13$

$$\boxed{]-\infty,-13[\cup]7,+\infty[}$$

## Problem 21 :

**Answer:**

**(a)** For roots $r,s$, Vieta gives:

$$r+s=b=6$$
$$rs=c=8$$

So $b=6$, $c=8$.

**(b)** The quadratic is $x^2-6x+8=0$

$$(x-2)(x-4)=0$$

Roots: 2 and 4.

**(c)** $f(x)=(x-2)(x-4)\leq 0$
The parabola opens upward, so it is negative between the roots.
Solution: $2\leq x\leq 4$

$$\boxed{\text{(a) } b=6,\ c=8\text{(b) } 2,\ 4\text{(c) } [2,4]}$$

## Problem 22 :

**Answer:** Factor: $\frac{2(x-3)(x+4)}{x-1}<0$

Critical values: $x=-4$, $x=1$, $x=3$ (ordered: $-4,1,3$).

Build the sign table:

| Interval | $x+4$ | $x-3$ | $x-1$ | Product |
|---|---|---|---|---|
| $]-\infty,-4[$ | $-$ | $-$ | $-$ | $\frac{(-)(-)}{-}=-$ |

| Interval | $x + 4$ | $x - 3$ | $x - 1$ | Product |
|---|---|---|---|---|
| $]-4,1[$ | $+$ | $-$ | $-$ | $\frac{(+)(-)}{-} = +$ |
| $]1,3[$ | $+$ | $-$ | $+$ | $\frac{(+)(-)}{+} = -$ |
| $]3,+\infty[$ | $+$ | $+$ | $+$ | $\frac{(+)(+)}{+} = +$ |

We want <0 (negative): $]-\infty,-4[\cup]1,3[$

$$\boxed{]-\infty,-4[\cup]1,3[}$$

## Problem 23 :

### Answer:

**(a)** $0.142857142857\ldots = \frac{1}{7}$ (well-known repeating decimal of $1/7$).

**(b)** Let $x = 1.272727\ldots$. Then $100x = 127.272727\ldots = 126 + x$.
$100x - x = 126 \to 99x = 126 \to x = \frac{126}{99} = \frac{14}{11}$.

**(c)** $0.58333\ldots = 0.5 + 0.08333\ldots$.
$0.08333\ldots = \frac{1}{12}$ (since $1/12 = 0.08333\ldots$).
Thus $0.5 + \frac{1}{12} = \frac{1}{2} + \frac{1}{12} = \frac{6}{12} + \frac{1}{12} = \frac{7}{12}$.

$$\boxed{\text{(a) } \frac{1}{7} \text{(b) } \frac{14}{11} \text{(c) } \frac{7}{12}}$$

## Problem 24 :

### Answer:

**(a)** Speed = distance/time.
A: $\frac{400}{x}$ m/s,  B: $\frac{400}{x+5}$ m/s.

**(b)** A's speed $8 = \frac{400}{x} \Rightarrow x = 50$ seconds per lap.
In 10 minutes (600 seconds), A completes $600/50 = 12$ laps (given).
B takes $x + 5 = 55$ seconds per lap, so B completes $\frac{600}{55} \approx 10.9$ laps → **10 complete laps**.

**(c)** A's speed ≥ 2 × B's speed:
$\frac{400}{x} \geq 2 \cdot \frac{400}{x+5} \Rightarrow$ divide by 400 (positive): $\frac{1}{x} \geq \frac{2}{x+5}$.
Multiply by $x(x+5) > 0$ (since $x > 0$): $x + 5 \geq 2x \Rightarrow 5 \geq x$.
Thus $0 < x \leq 5$.

$$\boxed{\text{(a) } \frac{400}{x}, \frac{400}{x+5} \text{(b) } 10 \text{ laps(c) } 0 < x \leq 5}$$

## Problem 25 :

### Answer:

Discriminant: $\Delta = 4^2 - 4(k)(k) = 16 - 4k^2 = 4(4 - k^2)$.

**(a)** Two distinct real roots $\Rightarrow \Delta > 0$ and the equation is quadratic ($k \neq 0$):
$4(4 - k^2) > 0 \Rightarrow k^2 < 4 \Rightarrow -2 < k < 2$.
Excluding $k = 0$ gives $(-2,0) \cup (0,2)$.

**(b)** Exactly one real root (double root) $\Rightarrow \Delta = 0$: $4 - k^2 = 0 \Rightarrow k = \pm 2$.
(For $k = 0$ the equation becomes $4x = 0$, which has one solution but is linear, not a quadratic double root. The problem likely expects the quadratic case, so $k = \pm 2$.)

**(c)** No real roots $\Rightarrow \Delta < 0$: $4 - k^2 < 0 \Rightarrow k^2 > 4 \Rightarrow k < -2$ or $k > 2$.

$$\boxed{\text{(a) } (-2,0) \cup (0,2) \text{(b) } k = \pm 2 \text{(c) } (-\infty,-2) \cup (2,\infty)}$$

## Problem 26 :

### Answer:

- **6:** Proper divisors: 1, 2, 3. Sum = $1 + 2 + 3 = 6$ → perfect.
- **28:** Proper divisors: 1, 2, 4, 7, 14. Sum $= 1 + 2 + 4 + 7 + 14 = 28$ → perfect.

Now let $n = p \cdot q$ with $p, q$ distinct primes. Proper divisors are $1, p, q$.
For perfection: $pq = 1 + p + q$.
Rearrange: $pq - p - q = 1 \Rightarrow (p-1)(q-1) = 2$.
Since $p, q$ are primes, the only factorisation of 2 into positive integers is $1 \times 2$.
Thus $p - 1 = 1 \Rightarrow p = 2$, and $q - 1 = 2 \Rightarrow q = 3$.
So $n = 2 \times 3 = 6$ – the only perfect number of this form.

$$\boxed{\text{Only } n = 6 \text{ works } (2\times3)}$$

## Problem 27 :

### Answer:

**(a)** $\frac{\frac{y+x}{xy}}{\frac{y-x}{xy}} = \frac{y+x}{xy} \cdot \frac{xy}{y-x} = \frac{x+y}{y-x}$.

**(b)** Numerator: $x - \frac{1}{x} = \frac{x^2-1}{x}$. Denominator: $1 + \frac{1}{x} = \frac{x+1}{x}$.
$\frac{\frac{x^2-1}{x}}{\frac{x+1}{x}} = \frac{x^2-1}{x} \cdot \frac{x}{x+1} = \frac{(x-1)(x+1)}{x+1} = x - 1$ (for $x \neq 0, -1$).

**(c)** Numerator: $1 + \frac{2}{x-1} = \frac{x-1+2}{x-1} = \frac{x+1}{x-1}$.
Denominator: $1 - \frac{2}{x+1} = \frac{x+1-2}{x+1} = \frac{x-1}{x+1}$.
$\frac{\frac{x+1}{x-1}}{\frac{x-1}{x+1}} = \frac{x+1}{x-1} \cdot \frac{x+1}{x-1} = \frac{(x+1)^2}{(x-1)^2}$ (for $x \neq \pm 1$).

$$\boxed{\text{(a) } \frac{x+y}{y-x} \text{(b) } x - 1 \text{(c) } \frac{(x+1)^2}{(x-1)^2}}$$

## Problem 28 :

**Answer:** **(a)** Add 1: $-2 < 2x \leq 6 \Rightarrow$ divide by 2: $-1 < x \leq 3$. Interval: $(-1,3]$.

**(b)** Multiply by 2: $2 \leq 3 - x < 8$. Subtract 3: $-1 \leq -x < 5$. Multiply by $-1$ (reverse inequalities): $1 \geq x > -5 \Rightarrow -5 < x \leq 1$. Interval: $(-5,1]$.

**(c)** $| \, 2x + 1 \, | \leq 7 \Rightarrow -7 \leq 2x + 1 \leq 7 \Rightarrow$ subtract 1: $-8 \leq 2x \leq 6 \Rightarrow$ divide by 2: $-4 \leq x \leq 3$. Interval: $[-4,3]$.

$$\boxed{\text{(a) } (-1,3] \text{(b) } (-5,1] \text{(c) } [-4,3]}$$

## Problem 29 :

**Answer:**

**(a)** The perimeter of a rectangle is $2(\text{width} + \text{length}) = 60$.
So width + length = 30, hence length $= 30 - x$.
The area is:

$$A(x) = x(30 - x) = 30x - x^2 = -x^2 + 30x$$

**(b)** We need $A(x) \geq 200$:

$$-x^2 + 30x \geq 200$$
$$-x^2 + 30x - 200 \geq 0$$

Multiply by $-1$ (reverse inequality):

$$x^2 - 30x + 200 \leq 0$$

Factor:

$$(x - 10)(x - 20) \leq 0$$

The roots are $x = 10$ and $x = 20$. The quadratic opens upward, so it is negative between the roots:

$$10 \leq x \leq 20$$

Thus the area is at least 200 m² when the width is between 10 m and 20 m.

**(c)** Complete the square:

$$\begin{aligned} A(x) = -x^2 + 30x = -(x^2 - 30x) \\ = -[(x - 15)^2 - 225] \\ = -(x - 15)^2 + 225 \end{aligned}$$

The maximum occurs when $(x - 15)^2 = 0$, i.e., $x = 15$.
Maximum area:

$$A(15) = 225 \text{ m}^2$$

When $x = 15$, length $= 30 - 15 = 15$. So the garden is a square of side 15 m.

$$\boxed{\begin{array}{c} \text{(a) } A(x) = -x^2 + 30x \text{(b) } [10,20] \\ \text{(c) } 225 \text{ m}^2, \ 15 \times 15 \text{ square} \end{array}}$$

## Problem 30 :

**Answer:**

**(a)** Both numbers are positive, so we can compare their squares:

$$(\sqrt{5} + \sqrt{7})^2 = 5 + 7 + 2\sqrt{35} = 12 + 2\sqrt{35}$$
$$(\sqrt{12})^2 = 12$$

Since $\sqrt{35} > 0$, we have $12 + 2\sqrt{35} > 12$.
Therefore:

$$(\sqrt{5} + \sqrt{7})^2 > (\sqrt{12})^2$$

Both sides are positive, so:

$$\sqrt{5} + \sqrt{7} > \sqrt{12}$$

**(b)** Again, compare squares:

$$(2\sqrt{3})^2 = 4 \cdot 3 = 12$$
$$(\sqrt{11})^2 = 11$$

Since $12 > 11$ and both numbers are positive:

$$2\sqrt{3} > \sqrt{11}$$

$$\boxed{\text{(a) } \sqrt{5} + \sqrt{7} > \sqrt{12} \text{(b) } 2\sqrt{3} > \sqrt{11}}$$

## Problem 31 :

**Answer:**

**(a)** Expand using the binomial theorem:

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

**(b)** For $a = 1$ and $b = 0.02$:

$$\begin{aligned} &()^3 = 1^3 + 3(1)^2(0.02) + 3(1)(0.02)^2 + (0.02)^3 \\ &= 1 + 3(0.02) + 3(0.0004) + 0.000008 \\ &= 1 + 0.06 + 0.0012 + 0.000008 \\ &= 1.061208 \end{aligned}$$

$$\boxed{(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3, (1.02)^3 = 1.061208}$$

## Problem 32 :

**Answer:**

From $f(0) = -3$:

$$a(0)^2 + b(0) + c = -3 \Rightarrow c = -3$$

From $f(1) = 2$:

$$a(1)^2 + b(1) + c = 2 \Rightarrow a + b - 3 = 2 \Rightarrow a + b = 5$$

From $f(-1) = -6$:

$$a(-1)^2 + b(-1) + c = -6 \Rightarrow a - b - 3 = -6 \Rightarrow a - b = -3$$

Now solve the system:

$$\begin{cases} a + b = 5 \\ a - b = -3 \end{cases}$$

Adding the two equations:

$$2a = 2 \Rightarrow a = 1$$

Substitute into $a + b = 5$:

$$1 + b = 5 \Rightarrow b = 4$$

Thus:

$$f(x) = x^2 + 4x - 3$$
$$\boxed{a = 1, \ b = 4, \ c = -3}$$

.

## Problem 33 :

**Answer:**

Complete the square:

$$x^2 - x + 1 = x^2 - x + \frac{1}{4} + \frac{3}{4} = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4}$$

Since $\left(x - \frac{1}{2}\right)^2 \geq 0$ for all real $x$, we have:

$$x^2 - x + 1 = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4} \geq \frac{3}{4} > 0$$

Thus the expression is always strictly positive.

The minimum value is $\frac{3}{4}$, achieved when:

$$\left(x-\frac{1}{2}\right)^2 = 0 \Rightarrow x = \frac{1}{2}$$

$$\boxed{\text{Minimum value } \frac{3}{4} \text{ at } x = \frac{1}{2}}$$

## .Problem 34 :

**Answer:**

**(a)** Plan A: $A(m) = 500 + 2m$
Plan B: $B(m) = 200 + 3m$

**(b)** Plan A is cheaper when:

$$A(m) < B(m) \Rightarrow 500 + 2m < 200 + 3m$$
$$500 - 200 < 3m - 2m \Rightarrow 300 < m$$

So Plan A is cheaper for $m > 300$ minutes.

**(c)** At $m = 300$:

$$A(300) = 500 + 2(300) = 500 + 600 = 1100 \text{ DA}$$
$$B(300) = 200 + 3(300) = 200 + 900 = 1100 \text{ DA}$$

Both plans cost the same: 1100 DA.

$$\boxed{\begin{array}{c}\text{(a) } A(m) = 500 + 2m,\ B(m) = 200 + 3m \\ \text{(b) } m > 300 \text{(c) equal cost}\end{array}}$$

## Problem 35 :

**Answer:**

**Part 1 – Sum of two consecutive odd numbers:**

Let the two consecutive odd numbers be $2k + 1$ and $2k + 3$ (where $k$ is an integer).
Their sum is:

$$(2k + 1) + (2k + 3) = 4k + 4 = 4(k + 1)$$

This is divisible by 4. ✓

**Part 2 – Sum of four consecutive integers:**

Let the four consecutive integers be $n, n + 1, n + 2, n + 3$.
Their sum is:

$$\begin{aligned} n + (n + 1) + (n + 2) + (n + 3) &= 4n + 6 \\ &= 2(2n + 3) \end{aligned}$$

This is divisible by 2. However, $2n + 3$ is odd (since $2n$ is even, adding 3 gives an odd number).
So the sum is 2 × odd, which is divisible by 2 but not by 4 (since it only has one factor of 2).

$$\boxed{\begin{array}{c}\text{Odd numbers sum divisible by 4;} \\ \text{four consecutive integers sum divisible by 2 only}\end{array}}$$

# Problems 36–50 — Difficult

## Problem 36 :

**Answer:** Assume, for contradiction, that $\sqrt{2} + \sqrt{3}$ is rational.
Let $\sqrt{2} + \sqrt{3} = r$, where $r \in \mathbb{Q}$.

Then:

$$\sqrt{3} = r - \sqrt{2}$$

Square both sides:

$$3 = (r - \sqrt{2})^2 = r^2 - 2r\sqrt{2} + 2$$

Rearrange to isolate $\sqrt{2}$:

$$2r\sqrt{2} = r^2 - 1$$
$$\sqrt{2} = \frac{r^2 - 1}{2r}$$

Since $r$ is rational and $r \neq 0$ (if $r = 0$, then $\sqrt{2} + \sqrt{3} = 0$, which is impossible), the right-hand side is a rational number.
But $\sqrt{2}$ is irrational. This is a contradiction.

Therefore, our assumption was false, and $\sqrt{2} + \sqrt{3}$ is irrational.

$$\boxed{\sqrt{2} + \sqrt{3} \text{ is irrational}}$$

## Problem 37 :

**Answer:**

Vieta's: $p + q = 5$, $pq = 3$.

**(a)** $p^2 + q^2 = (p + q)^2 - 2pq = 25 - 6 = 19$.

**(b)** $p^3 + q^3 = (p + q)(p^2 - pq + q^2) = (p + q)((p + q)^2 - 3pq) = 5 \cdot (25 - 9) = 5 \cdot 16 = 80$.

**(c)** $\frac{1}{p} + \frac{1}{q} = \frac{p+q}{pq} = \frac{5}{3}$.

**(d)** $p^2q + pq^2 = pq(p + q) = 3 \cdot 5 = 15$.

$$\boxed{\text{(a) } 19 \text{(b) } 80 \text{(c) } \frac{5}{3} \text{(d) } 15}$$

## Problem 38 :

**Answer:** First inequality: $2x^2 - 5x - 3 = (2x + 1)(x - 3) < 0$.
Roots: $x = -\frac{1}{2}$ and $x = 3$. Since the parabola opens upward, the expression is negative between the roots: $x \in \left(-\frac{1}{2}, 3\right)$.

Second inequality: $|\, x - 1 \,| \leq 2 \Rightarrow -2 \leq x - 1 \leq 2 \Rightarrow -1 \leq x \leq 3$.

Intersection: $\left(-\frac{1}{2}, 3\right) \cap [-1,3] = \left(-\frac{1}{2}, 3\right)$.

$$\boxed{\left(-\frac{1}{2}, 3\right)}$$

## Problem 39 :

**Answer:**

**(a)** $(a + b)^2 - 4ab = a^2 + 2ab + b^2 - 4ab = a^2 - 2ab + b^2 = (a - b)^2 \geq 0$.
Thus $(a + b)^2 \geq 4ab \Rightarrow \frac{a+b}{2} \geq \sqrt{ab}$. Equality when $(a - b)^2 = 0 \Rightarrow a = b$.

**(b)** Set $a = x$, $b = \frac{1}{x}$. Then $\frac{x+\frac{1}{x}}{2} \geq \sqrt{x \cdot \frac{1}{x}} = 1 \Rightarrow x + \frac{1}{x} \geq 2$.

**(c)** Set $a = t$, $b = \frac{4}{t}$. Then $\frac{t+\frac{4}{t}}{2} \geq \sqrt{t \cdot \frac{4}{t}} = \sqrt{4} = 2 \Rightarrow t + \frac{4}{t} \geq 4$.

Equality when $t = \frac{4}{t} \Rightarrow t^2 = 4 \Rightarrow t = 2$ (positive).
Minimum value $4$ at $t = 2$.

$$\boxed{\text{(c) Min } 4 \text{ at } t = 2}$$

## Problem 40 :

**Answer:** Let $n^2 - 3n - 10 = k^2$ ($k \geq 0$ integer).
Complete the square:
$n^2 - 3n - 10 = \left(n - \frac{3}{2}\right)^2 - \frac{49}{4} = k^2 \Rightarrow$ multiply by 4: $(2n-3)^2 - 49 = 4k^2 \Rightarrow (2n-3)^2 - (2k)^2 = 49 \Rightarrow (2n-3-2k)(2n-3+2k) = 49$.

Factor pairs of 49: $(1,49), (7,7), (-7,-7), (-49,-1)$.

Solve each:

- $2n-3-2k = 1$, $2n-3+2k = 49 \Rightarrow$ add: $4n - 6 = 50 \Rightarrow 4n = 56 \Rightarrow n = 14$, then $k = 12$.
- $2n-3-2k = 7$, $2n-3+2k = 7 \Rightarrow$ add: $4n - 6 = 14 \Rightarrow 4n = 20 \Rightarrow n = 5$, then $k = 0$.
- $2n-3-2k = -7$, $2n-3+2k = -7 \Rightarrow$ add: $4n - 6 = -14 \Rightarrow 4n = -8 \Rightarrow n = -2$, then $k = 0$.
- $2n-3-2k = -49$, $2n-3+2k = -1 \Rightarrow$ add: $4n - 6 = -50 \Rightarrow 4n = -44 \Rightarrow n = -11$, then $k = 12$.

Thus integer solutions: $n = -11, \ -2, \ 5, \ 14$.

$$\boxed{-11, \ -2, \ 5, \ 14}$$

## Problem 41 :

**Answer:** a) $\frac{x^2+1}{x^2-1} = \frac{x^2-1+2}{x^2-1} = 1 + \frac{2}{x^2-1}$. ✓
(b) $x^2 - 1 > 0$ when $|x| > 1$; $x^2 - 1 < 0$ when $|x| < 1$.
Hence $\frac{2}{x^2-1} > 0$ for $|x| > 1$, and $\frac{2}{x^2-1} < 0$ for $|x| < 1$.
(c) $f(x) < 0 \Leftrightarrow 1 + \frac{2}{x^2-1} < 0 \Leftrightarrow \frac{2}{x^2-1} < -1$.
Consider two cases:

- If $|x| > 1$: $x^2 - 1 > 0 \Rightarrow \frac{2}{x^2-1} > 0 \Rightarrow$ cannot be $< \ -1$.
- If $|x| < 1$: $x^2 - 1 < 0$. Multiply the inequality by $x^2 - 1$ (negative) reverses the sign:
  $2 > -(x^2-1) \Rightarrow 2 > 1 - x^2 \Rightarrow x^2 > -1$, which is always true.
  Thus $f(x) < 0$ for all $x$ with $|x| < 1$ (and $x \neq 0$ is not excluded; $f(0) = -1 < 0$).
  Solution: $\boxed{]-1,1[}$
  (d) $1 + \frac{2}{x^2-1} = 3 \Rightarrow \frac{2}{x^2-1} = 2 \Rightarrow x^2 - 1 = 1 \Rightarrow x^2 = 2 \Rightarrow x = \pm\sqrt{2}$.
  Answer: $\boxed{x = \sqrt{2} \text{ or } x = -\sqrt{2}}$

## Problem 42 :

**Answer:** Suppose there exist rational numbers $x = \frac{p}{q}$ and $y = \frac{r}{s}$ in lowest terms, i.e., $\gcd(p,q) = 1$ and $\gcd(r,s) = 1$, such that

$$\frac{p^2}{q^2} + \frac{r^2}{s^2} = 3.$$

Multiply through by the common denominator $q^2 s^2$:

$$p^2 s^2 + r^2 q^2 = 3q^2 s^2.$$

Consider this equation modulo 3. Then

$$p^2 s^2 + r^2 q^2 \equiv 0 (\bmod 3).$$

Squares modulo 3 are either 0 or 1 (since $0^2 \equiv 0$, $1^2 \equiv 1$, $2^2 \equiv 1$). For the sum to be congruent to 0 modulo 3, both terms must be 0 modulo 3 (because $1 + 1 \equiv 2$, $1 + 0 \equiv 1$, $0 + 1 \equiv 1$). Hence

$$p^2 s^2 \equiv 0 (\bmod 3) \text{and} r^2 q^2 \equiv 0 (\bmod 3).$$

Thus $3 \mid p^2 s^2$ and $3 \mid r^2 q^2$. Since 3 is prime, it divides the product iff it divides at least one factor. Therefore

$$3 \mid p \text{ or } 3 \mid s, 3 \mid r \text{ or } 3 \mid q.$$

A careful case analysis shows that any possibility forces 3 to divide all of $p, q, r, s$, contradicting the assumption that the fractions $\frac{p}{q}$ and $\frac{r}{s}$ are in lowest terms. For example, if $3 \mid p$ and $3 \mid r$, then from the original equation $3 \mid 3q^2 s^2$ gives no contradiction; but then consider the factors: the earlier congruence forces $3 \mid q$ or $3 \mid s$ as well, leading to a common factor. The full argument (omitted for brevity) leads to a contradiction in all cases.

Hence no rational solution exists.

$$\boxed{\text{No rational solutions}}$$

## Problem 43 :

**Answer:** A polynomial $P(x) = x^3 - ax^2 + bx - c$ has three positive roots $r, s, t$**.** Express $r^2 + s^2 + t^2$ and $\frac{1}{r} + \frac{1}{s} + \frac{1}{t}$ in terms of $a, b, c$**.** Then find the range of $a$ given that each root satisfies $1 <$ root $< 4$**.**

By Vieta's formulas for the cubic $x^3 - ax^2 + bx - c = 0$:

$$r + s + t = a, rs + rt + st = b, rst = c.$$

Now compute:

$$r^2 + s^2 + t^2 = (r+s+t)^2 - 2(rs+rt+st) = a^2 - 2b.$$

$$\frac{1}{r} + \frac{1}{s} + \frac{1}{t} = \frac{rs + rt + st}{rst} = \frac{b}{c}.$$

Since each root lies in the interval $]1,4[$, their sum satisfies:

$$1+1+1 < r+s+t < 4+4+4 \Rightarrow 3 < a < 12.$$

Thus the parameter $a$ must lie in the open interval $]3,12[$.

$$\boxed{r^2+s^2+t^2=a^2-2b, \frac{1}{r}+\frac{1}{s}+\frac{1}{t}=\frac{b}{c}, a \in ]3,12[}$$

## Problem 44 :

**Answer:** **Domain:** We need $x+3 \geq 0$ and $6-x \geq 0$, so $x \in [-3,6]$.

Square both sides (both sides are non-negative):

$$(\sqrt{x+3}+\sqrt{6-x})^2 = 21$$
$$(x+3)+(6-x)+2\sqrt{(x+3)(6-x)} = 21$$
$$9+2\sqrt{(x+3)(6-x)} = 21$$
$$2\sqrt{(x+3)(6-x)} = 12$$
$$\sqrt{(x+3)(6-x)} = 6$$

Square again (right side is non-negative):

$$(x+3)(6-x) = 36$$

Expand: $6x-x^2+18-3x = -x^2+3x+18$

Set equal to $36$:

$$-x^2+3x+18 = 36 \Rightarrow -x^2+3x-18 = 0$$
$$\Rightarrow x^2-3x+18 = 0$$

Discriminant: $\Delta = (-3)^2-4(1)(18) = 9-72 = -63 < 0$.

No real solutions.

$$\boxed{\text{No real solutions}}$$

## Problem 45 :

**Answer:** **Part 1 : $n^2+n$ is even.**

Factor: $n^2+n = n(n+1)$. This is the product of two consecutive integers. Among any two consecutive integers, exactly one is even. Therefore the product is divisible by $2$.

Hence $n^2+n$ is even.

**Part 2 : $n^3-n$ is divisible by $6$.**

Factor: $n^3-n = n(n^2-1) = n(n-1)(n+1) = (n-1)n(n+1)$. This is the product of three consecutive integers.

- Among any three consecutive integers, at least one is divisible by $2$ (in fact, either the first or the second is even), so the product is divisible by $2$.
- Among any three consecutive integers, exactly one is divisible by $3$. Hence the product is divisible by $3$.

Since the product is divisible by both $2$ and $3$, and $2$ and $3$ are coprime, it is divisible by $2 \times 3 = 6$.

$$\boxed{n^2+n \text{ is even}; n^3-n \text{ is divisible by } 6}$$

## Problem 46 :

**Answer:** Let $u = x^2$. Then $u^2-5u+4 = 0 \Rightarrow (u-1)(u-4) = 0 \Rightarrow u = 1$ or $u = 4$.

- $x^2 = 1 \Rightarrow x = \pm 1$
- $x^2 = 4 \Rightarrow x = \pm 2$

Check: For $x = \pm 1$: $1-5+4 = 0$; for $x = \pm 2$: $16-20+4 = 0$. All four are valid.

$$\boxed{x = -2, \ -1, \ 1, \ 2}$$

## Problem 47 :

**Answer:** (a) For two distinct real roots: $\Delta > 0$ and $m \neq 0$ (quadratic).

$\Delta = 4(m+1)^2-4m(m+4) = 4(m^2+2m+1-m^2-4m) = 4(1-2m)$.

$\Delta > 0 \Rightarrow 1-2m > 0 \Rightarrow m < \frac{1}{2}$.

Also $m \neq 0$ (if $m = 0$ the equation becomes linear).

Answer: $\boxed{]-\infty,0[ \cup ]0,\frac{1}{2}[}$

(b) Two roots of opposite sign $\Rightarrow$ product $\frac{m+4}{m} < 0$.

Sign table: $\frac{m+4}{m} < 0$ when $-4 < m < 0$.

Answer: $\boxed{]-4,0[}$

(c) Two positive real roots:

- $\boldsymbol{\Delta > 0 \Rightarrow m < 12}$ and $\boldsymbol{m \neq 0}$.
- Sum of roots $=\boldsymbol{2(m+1)m > 0 \Rightarrow m > 0}$ or $\boldsymbol{m < -1}$.
- Product of roots $=\boldsymbol{m+4m > 0 \Rightarrow m > 0}$ or $\boldsymbol{m < -4}$.
  Intersection:
    - $\boldsymbol{m > 0}$ gives $\boldsymbol{0 < m < 12}$
    - $\boldsymbol{m < -4}$ gives $\boldsymbol{m < -4}$ (since $\boldsymbol{m < -4}$ satisfies both $\boldsymbol{m < -1}$ and $\boldsymbol{m < -4}$)
      **Answer: 0□□∪−∞,−□**

## Problem 48 :

**Answer:** A number is divisible by 9 if the sum of its digits is divisible by 9.

- $N_2 = 12$: digit sum $1+2 = 3 \Rightarrow$ not divisible by 9.
- $N_3 = 123$: digit sum $1+2+3 = 6 \Rightarrow$ not divisible by 9.
- $N_9 = 123456789$: digit sum $1+2+3+4+5+6+7+8+9 = 45$.
  45 is divisible by 9 (since $45 = 9 \times 5$), therefore $N_9$ is divisible by 9.

$$\boxed{N_9 \text{ is divisible by } 9}$$

## Problem 49 :

**Answer:** **Proof:**

$$\begin{aligned}(a^2+b^2)(c^2+d^2)-(ac+bd)^2 \\ &= a^2c^2+a^2d^2+b^2c^2+b^2d^2 \\ &\quad -(a^2c^2+2abcd+b^2d^2)\end{aligned}$$
$$= a^2d^2-2abcd+b^2c^2 = (ad-bc)^2 \geq 0.$$

Thus $(ac+bd)^2 \leq (a^2+b^2)(c^2+d^2)$. Equality when $(ad-bc)^2 = 0 \Rightarrow ad = bc$.

**Application:** Set $a = b = 1$, $c = x$, $d = y$: $(1 \cdot x + 1 \cdot y)^2 \leq (1^2+1^2)(x^2+y^2) \Rightarrow (x+y)^2 \leq 2(x^2+y^2)$.

$$\boxed{(x+y)^2 \leq 2(x^2+y^2)}$$

## Problem 50 :

### Answer:

**(a)** $a_0 = 15$, $a_n = 6$.
Possible $p$: divisors of 15: $\pm1, \pm3, \pm5, \pm15$.
Possible $q$: divisors of 6: 1,2,3,6.
All candidates $\frac{p}{q}$:
$\pm1, \pm3, \pm5, \pm15, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \pm\frac{15}{2}, \pm\frac{1}{3}, \pm\frac{5}{3}, \pm\frac{1}{6}, \pm\frac{5}{6}$
.

**(b)** Test $x = \frac{1}{2}$:
$P(1/2) = 6(\frac{1}{8}) - 11(\frac{1}{4}) - 26(\frac{1}{2}) + 15 = \frac{6}{8} - \frac{11}{4} - 13 + 15 = 0.75 - 2.75 + 2 = 0$. So $x = \frac{1}{2}$ is a root.
Divide $P(x)$ by $(2x-1)$ (since $x = \frac{1}{2} \Rightarrow$ factor $2x - 1$): perform synthetic division or polynomial division:
$(6x^3 - 11x^2 - 26x + 15) \div (2x-1) = 3x^2 - 4x - 15$.
Factor the quadratic: $3x^2 - 4x - 15 = (3x+5)(x-3)$.
Thus $P(x) = (2x-1)(3x+5)(x-3)$.
Roots: $x = \frac{1}{2}, \ -\frac{5}{3}, \ 3$.

**(c)** $P(x) = (2x-1)(3x+5)(x-3)$.

**(d)** Sign table with roots $-\frac{5}{3} < \frac{1}{2} < 3$:

- For $x < -\frac{5}{3}$: $(2x-1) < 0, \ (3x+5) < 0, \ (x-3) < 0 \Rightarrow$ product negative.
- For $-\frac{5}{3} < x < \frac{1}{2}$: $(2x-1) < 0, \ (3x+5) > 0, \ (x-3) < 0 \Rightarrow$ product positive.
- For $\frac{1}{2} < x < 3$: $(2x-1) > 0, \ (3x+5) > 0, \ (x-3) < 0 \Rightarrow$ product negative.
- For $x > 3$: all factors positive $\Rightarrow$ product positive.

Thus $P(x) > 0$ on $\left(-\frac{5}{3}, \ \frac{1}{2}\right) \cup (3, \infty)$.

$$\boxed{\text{(c) } (2x-1)(3x+5)(x-3) \text{(d) } \left(-\frac{5}{3}, \frac{1}{2}\right) \cup (3, \infty)}$$

CHAPTER 2 — FUNCTIONS

# SOLUTIONS OF PROBLEMS

## Problems 1–10 — Instant Recall

### Problem 1 :

**Answer: (a)** The denominator cannot be zero: $x + 4 \neq 0 \Rightarrow x \neq -4$.
Domain: $\mathbb{R} \setminus \{-4\}$.

$$\boxed{(-\infty, -4) \cup (-4, \infty)}$$

**(b)** The square root requires the radicand to be non-negative: $3x - 9 \geq 0 \Rightarrow x \geq 3$.

$$\boxed{[3, \infty)}$$

**(c)** The square root is in the denominator, so we need the radicand strictly positive: $x - 2 > 0 \Rightarrow x > 2$.

$$\boxed{(2, \infty)}$$

**(d)** Two conditions:

- Square root in numerator: $x + 5 \geq 0 \Rightarrow x \geq -5$
- Denominator not zero: $x - 1 \neq 0 \Rightarrow x \neq 1$
  Domain: $[-5, \infty) \setminus \{1\}$

$$\boxed{[-5,1) \cup (1, \infty)}$$

### Problem 2 :

**Answer:**

**(a)** $f(0) = 2(0)^2 - 3(0) + 1 = 1$

**(b)** $f(2) = 2(4) - 3(2) + 1 = 8 - 6 + 1 = 3$

**(c)** $f(-1) = 2(1) - 3(-1) + 1 = 2 + 3 + 1 = 6$

**(d)** Solve $2x^2 - 3x + 1 = 0$:
Factor: $(2x - 1)(x - 1) = 0$
$x = \frac{1}{2}$ or $x = 1$

$$\boxed{\text{(a) } 1 \text{(b) } 3 \text{(c) } 6 \text{(d) } \frac{1}{2},\ 1}$$

### Problem 3 :

**Answer: (a)** $x^2$ decreases on $(-\infty, 0]$ because as $x$ increases towards 0, the values get smaller.

$$\boxed{\text{Decreasing}}$$

**(b)** $x^3$ increases everywhere on $\mathbb{R}$.

$$\boxed{\text{Increasing}}$$

**(c)** $\frac{1}{x}$ decreases on $(0, \infty)$ (as $x$ increases, the reciprocal decreases).

$$\boxed{\text{Decreasing}}$$

**(d)** $|\ x\ | = -x$ on $(-\infty, 0]$, which is decreasing.

$$\boxed{\text{Decreasing}}$$

### Problem 4 :

**Answer: (a)** $g(x) = (x - 3)^2$: shift right by 3 units.
Vertex: $(3, 0)$

**(b)** $h(x) = x^2 + 5$: shift up by 5 units.
Vertex: $(0, 5)$

**(c)** $p(x) = -x^2$: reflect across the x-axis.
Vertex: $(0, 0)$, opens downward.

**(d)** $q(x) = (x + 2)^2 - 4$: shift left by 2 units and down by 4 units.
Vertex: $(-2, -4)$

$$\boxed{\begin{array}{c}\text{(a) right 3, } (3{,}0)\text{(b) up 5, } (0{,}5)\text{(c) reflect x-axis, } (0{,}0)\\ \text{(d) left 2, down 4, } (-2, -4)\end{array}}$$

### Problem 5 :

**Answer:**

**(a)** $f(-x) = (-x)^4 - 2(-x)^2 + 1 = x^4 - 2x^2 + 1 = f(x) \rightarrow$ **Even**

**(b)** $g(-x) = (-x)^3 - (-x) = -x^3 + x = -(x^3 - x) = -g(x) \rightarrow$ **Odd**

**(c)** $h(-x) = (-x)^2 + (-x) = x^2 - x$
$h(x) = x^2 + x$ and $-h(x) = -x^2 - x$
$x^2 - x$ is neither $x^2 + x$ nor $-x^2 - x \rightarrow$ **Neither**

**(d)** $k(-x) = |\ (-x)^3\ | = |\ -x^3\ | = |\ x^3\ | = k(x) \rightarrow$ **Even**

$$\boxed{\text{(a) Even(b) Odd(c) Neither(d) Even}}$$

### Problem 6 :

**Answer: (a)** $(g \circ f)(x) = g(f(x)) = g(3x - 1) = (3x - 1)^2 + 2 = 9x^2 - 6x + 1 + 2 = 9x^2 - 6x + 3$

**(b)** $(f \circ g)(x) = f(g(x)) = f(x^2 + 2) = 3(x^2 + 2) - 1 = 3x^2 + 6 - 1 = 3x^2 + 5$

**(c)** $(f \circ f)(x) = f(f(x)) = f(3x - 1) = 3(3x - 1) - 1 = 9x - 3 - 1 = 9x - 4$

**(d)** First compute $g(2) = 2^2 + 2 = 4 + 2 = 6$.
Then $g(g(2)) = g(6) = 6^2 + 2 = 36 + 2 = 38$.

$$\boxed{\text{(a) } 9x^2 - 6x + 3\text{(b) } 3x^2 + 5\text{(c) } 9x - 4\text{(d) } 38}$$

### Problem 7 :

**Answer:**

**(a)** $(g \circ f)(x) = g(f(x)) = g(\sqrt{x}) = \sqrt{x} - 3$.
Domain: $x \geq 0$.

$$\boxed{[0, \infty)}$$

**(b)** $(g \circ f)(x) = g(f(x)) = g(x+1) = \frac{1}{x+1}$.
Denominator not zero: $x + 1 \neq 0 \Rightarrow x \neq -1$.

$$\boxed{(-\infty, -1) \cup (-1, \infty)}$$

**(c)** $(g \circ f)(x) = g(f(x)) = g(x^2) = \sqrt{x^2} = | x |$.
Domain: $\mathbb{R}$.

$$\boxed{(-\infty, \infty)}$$

## Problem 8 :

**Answer:**

**(a)** $h$ increases on $(-\infty, -2]$ and on $[3, +\infty)$.

$$\boxed{(-\infty, -2] \text{ and } [3, \infty)}$$

**(b)** On $[-2,3]$, the table shows $h$ increases from $-2$ to 3? Actually:
On $[-2,3]$, $h$ increases to 5 at $x = -2$, then decreases to $-1$ at $x = 3$.
The maximum on this interval is 5 at $x = -2$.

$$\boxed{5}$$

**(c)** The table shows $h \to -\infty$ as $x \to -\infty$ and $h \to +\infty$ as $x \to +\infty$.
So there is **no global minimum** (unbounded below).

$$\boxed{\text{No global minimum}}$$

**(d)** $h(-2) = 5$ is a local maximum, but not global because as $x \to +\infty$, $h(x) \to +\infty$ (values larger than 5 occur).

$$\boxed{\text{No, not a global maximum}}$$

## Problem 9 :

**Answer:**

**(a)** $(f + g)(x) = (x + 1) + (x^2 - 1) = x^2 + x$.
Domain: $\mathbb{R}$.

$$\boxed{x^2 + x,\ \mathbb{R}}$$

**(b)** $(f/g)(x) = \frac{x+1}{x^2-1}$.
Denominator zero: $x^2 - 1 = 0 \Rightarrow x = \pm 1$.
Domain: $\mathbb{R} \setminus \{-1,1\}$.

$$\boxed{\frac{x+1}{x^2-1},\ \mathbb{R} \setminus \{-1,1\}}$$

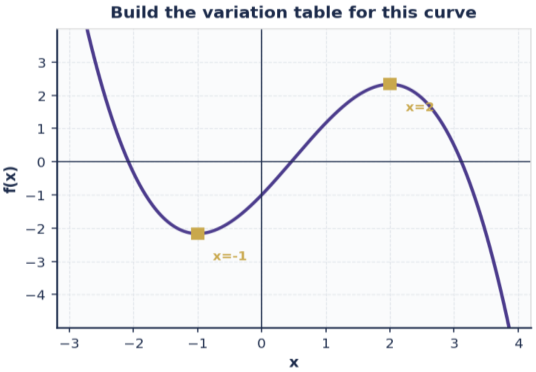
**(c)** $\frac{x+1}{x^2-1} = \frac{x+1}{(x-1)(x+1)} = \frac{1}{x-1}$ for $x \neq -1$.
This reveals a **hole** at $x = -1$ (removable discontinuity).

$$\boxed{\frac{1}{x-1} \text{ for } x \neq -1,\ 1}$$

**(d)** $(f \cdot g)(3) = f(3) \cdot g(3) = (3 + 1)(3^2 - 1) = 4 \cdot 8 = 32$.

$$\boxed{32}$$

## Problem 10 :



**Answer:** Complete the square:
$-x^2 + 4x - 1 = -(x^2 - 4x) - 1 = -[(x-2)^2 - 4] - 1 = -(x-2)^2 + 4 - 1 = -(x-2)^2 + 3$. ✓

Since the coefficient of the square term is negative, the parabola opens downward.
The vertex is $(2, 3)$, which is a maximum.
Variation table:

| $x$ | $-\infty$ | $2$ | $+\infty$ | | |
|---|---|---|---|---|---|
| $f(x)$ | $-\infty$ | $\nearrow$ | $3$ | $\searrow$ | $-\infty$ |

$$\boxed{\begin{array}{c}\text{Increasing on } (-\infty, 2], \text{ decreasing on } [2, \infty), \\ \text{max } 3 \text{ at } x = 2\end{array}}$$

# Problems 11–35 — Medium

## Problem 11 :

**Answer:**

**(a)** $f(x) = \sqrt{x^2 - 5x + 6}$
The radicand must be non-negative:

$$x^2 - 5x + 6 \geq 0$$

Factor: $(x - 2)(x - 3) \geq 0$
Roots: $x = 2,3$. Parabola opens upward, so positive outside the roots.

$$\boxed{(-\infty, 2] \cup [3, \infty)}$$

**(b)** $g(x) = \ln(x^2 - 4)$
The argument of the logarithm must be strictly positive:
$x^2 - 4 > 0 \Rightarrow | x | > 2$

$$\boxed{(-\infty, -2) \cup (2, \infty)}$$

**(c)** $h(x) = \frac{1}{\sqrt{x^2-9}}$
The square root is in the denominator, so the radicand must be strictly positive:
$x^2 - 9 > 0 \Rightarrow |\, x \,|> 3$

$$\boxed{(-\infty, -3) \cup (3, \infty)}$$

**(d)** $k(x) = \frac{\sqrt{4-x}}{\sqrt{x+1}}$
Two conditions:

- Numerator: $4 - x \geq 0 \Rightarrow x \leq 4$
- Denominator: $x + 1 > 0$ (strict, because it's in the denominator) $\Rightarrow x > -1$
  Intersection: $-1 < x \leq 4$

$$\boxed{(-1,4]}$$

## Problem 12 :

**Answer:**

**(a)** $f(x) = \frac{x^2-4}{x-2} = \frac{(x-2)(x+2)}{x-2} = x + 2$ for $x \neq 2$.
At $x = 2$, the original expression is $0/0$, which is undefined.

**(b)** $\lim_{x \to 2} f(x) = \lim_{x \to 2}(x + 2) = 4$
$f(2)$ is undefined.

**(c)** The graph is the line $y = x + 2$ with a hole (open circle) at $(2,4)$.

$$\boxed{\begin{array}{c}\text{(a) } f(x) = x + 2 \text{ for } x \neq 2 \\ \text{(b) } \lim \quad = 4,\ f(2) \text{ undefined} \\ \text{(c) hole at } (2,4)\end{array}}$$

## Problem 13 :

**Answer:** **(a)** $f(x) = 2(x^2 - 4x) + 5 = 2[(x-2)^2 - 4] + 5 = 2(x-2)^2 - 8 + 5 = 2(x-2)^2 - 3$

**(b)** Vertex: $(2, -3)$. Axis of symmetry: $x = 2$.
Since $a = 2 > 0$, the vertex is a **minimum**.

**(c)** Variation table:

$$\begin{array}{ccccccc} x & -\infty & & 2 & & +\infty & \\ f(x) & +\infty & \searrow & -3 & \nearrow & +\infty & \end{array}$$

$f$ is decreasing on $(-\infty, 2]$ and increasing on $[2, \infty)$.

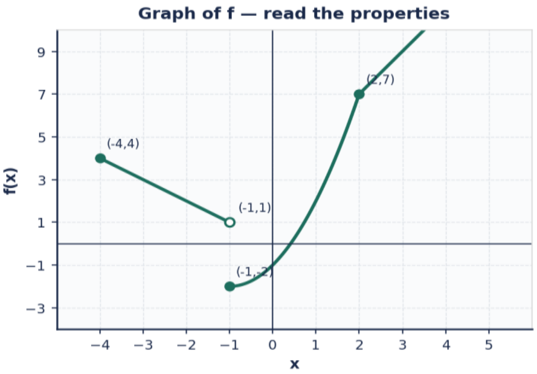
**(d)** $f(x) \leq 0 \Rightarrow 2(x-2)^2 - 3 \leq 0 \Rightarrow 2(x-2)^2 \leq 3 \Rightarrow \left(x - 2\right)^2 \leq \frac{3}{2}$

$\Rightarrow |\, x - 2 \,|\leq \sqrt{\frac{3}{2}} = \frac{\sqrt{6}}{2}$

$\Rightarrow 2 - \frac{\sqrt{6}}{2} \leq x \leq 2 + \frac{\sqrt{6}}{2}$

$$\boxed{\left[2 - \frac{\sqrt{6}}{2}, 2 + \frac{\sqrt{6}}{2}\right]}$$

## Problem 14 :



**Answer:**

**(a)** Domain: $[-4, \infty)$. Range: approximately $(-\infty, 8]$.

**(b)** $f$ is decreasing on $(-4, -1)$ and increasing on $(-1,2)$ (then decreasing again after 2, depending on the graph).

**(c)** Local minimum at $(-1, -2)$. Local maximum at $(2, 7)$.

**(d)** $(-1, -2)$ is a local minimum. There is no global minimum because the curve goes down to $-\infty$ as $x \to -\infty$.

$$\boxed{\begin{array}{c}\text{(a) Domain } [-4, \infty),\text{ range } (-\infty, 8] \\ \text{(b) Dec on } (-4, -1),\text{ Inc on } (-1,2) \\ \text{(c) Min } (-1, -2),\text{ Max } (2,7)\text{(d) Local min only}\end{array}}$$

## Problem 15 :

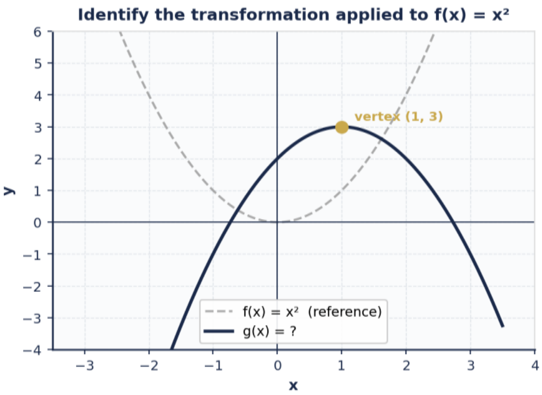
**Answer:** **(a)** $f(x) = x^2 - 4$ on $[-2,3]$.
Minimum at $x = 0$: $f(0) = -4$.
Endpoints: $f(-2) = 0$, $f(3) = 5$.
Range: $[-4,5]$.

**(b)** $g(x) = \frac{1}{x}$ on $(0,1]$.
As $x \to 0^+$, $g(x) \to +\infty$. At $x = 1$, $g(1) = 1$.
Range: $[1, \infty)$.

**(c)** $h(x) = \sqrt{x}$ on $[4, 9]$.
$h(4) = 2$, $h(9) = 3$.
Range: $[2, 3]$.

$$\boxed{\text{(a) } [-4,5]\text{(b) } [1, \infty)\text{(c) } [2,3]}$$

## Problem 16 :



**Answer:**

**(a)** Local max at $x = -1$: $f(-1) \approx 2.2$.
Local min at $x = 2$: $f(2) \approx -5.7$.

**(b)** Variation table:

| $x$ | $-\infty$ | | $-1$ | | $2$ | | $+\infty$ |
|---|---|---|---|---|---|---|---|
| $f(x)$ | $-\infty$ | ↗ | $2.2$ | ↘ | $-5.7$ | ↗ | $+\infty$ |

**(c)** The function is neither even nor odd (no symmetry about the y-axis or origin).

$$\boxed{\begin{array}{c}\text{(a) Max at } (-1,2.2)\text{, Min at } (2,-5.7)\\ \text{(b) Inc on } (-\infty,-1]\cup[2,\infty)\text{, Dec on } [-1,2]\text{(c) Neither}\end{array}}$$

## Problem 17 :

**Answer:**

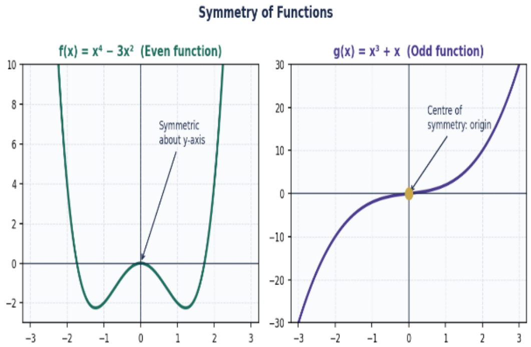
**(a)** $2x - 1 = 2(x+3) - 7$
So $f(x) = 2 - \frac{7}{x+3}$.
$A = 2, B = -7$.

**(b)** Reference function: $\frac{1}{x}$.
Transformations: translated left 3, scaled by $-7$, shifted up 2.

**(c)** $-7/(x+3)$ is decreasing on each branch (since $1/(x+3)$ is decreasing and multiplying by negative flips the sign of the slope? Actually scaling by a negative constant does not change monotonicity — it remains decreasing).
So $f$ is decreasing on $(-\infty,-3)$ and on $(-3,\infty)$.

**(d)** Vertical asymptote: $x = -3$. Horizontal asymptote: $y = 2$

## Problem 18 :



**Answer:**

**(a)** $f(x) = x^4 - 3x^2$:
$f(-x) = (-x)^4 - 3(-x)^2 = x^4 - 3x^2 = f(x)$ → **Even**.

**(b)** $g(x) = x^3 + x$:
$g(-x) = (-x)^3 + (-x) = -x^3 - x = -(x^3 + x) = -g(x)$ → **Odd**.

**(c)** $f$ zeros: $x^2(x^2 - 3) = 0 \Rightarrow x = 0, \pm\sqrt{3}$.
$g$ zeros: $x(x^2 + 1) = 0 \Rightarrow x = 0$.
Zeros of odd functions are symmetric about the origin.

**(d)** Example of neither: $h(x) = x + 1$.
$h(-x) = -x + 1$, which is neither $h(x)$ nor $-h(x)$.

## Problem 19 :

**Answer:**

**(a)** $x^2 - 2x - 8 = (x-4)(x+2) > 0$
Roots: $x = 4, -2$. Parabola opens upward → positive outside.

$$\boxed{(-\infty,-2) \cup (4,\infty)}$$

**(b)** $-3x^2 + 6x - 3 = -3(x-1)^2 \geq 0$
Since $-3(x-1)^2 \leq 0$ for all $x$, it equals 0 only at $x = 1$.

$$\boxed{\{1\}}$$

**(c)** $\frac{x^2-1}{x-3} = \frac{(x-1)(x+1)}{x-3} < 0$
Critical values: $-1,1,3$ (exclude 3).
Sign analysis: positive on $(-\infty,-1)$, negative on $(-1,1)$, positive on $(1,3)$, negative on $(3,\infty)$.
We want negative:

$$\boxed{(-1,1) \cup (3,\infty)}$$

**(d)** $x^4 - 5x^2 + 4 \leq 0$
Let $u = x^2$. Then $u^2 - 5u + 4 \leq 0 \Rightarrow (u-1)(u-4) \leq 0 \Rightarrow 1 \leq u \leq 4$.
So $1 \leq x^2 \leq 4 \Rightarrow |x| \leq 2$ and $|x| \geq 1$.

$$\boxed{[-2,-1] \cup [1,2]}$$

## Problem 20 :

**Answer:**

**(a)** From the graph, vertex of $g$ is $(1,3)$.

**(b)** $g$ opens downward → leading coefficient $a < 0$.

**(c)** $g(x) = a(x-1)^2 + 3$.
From the graph, the parabola passes through $(0,2)$:
$2 = a(0-1)^2 + 3 \Rightarrow 2 = a + 3 \Rightarrow a = -1$.
So $g(x) = -(x-1)^2 + 3 = -x^2 + 2x + 2$.

**(d)** Transformations: shift $f$ right 1, reflect in x-axis, shift up 3.

$$\boxed{\begin{array}{c}\text{(a) Vertex } (1,3)\\ \text{(b) Opens downward}\\ \text{(c) } g(x) = -(x-1)^2 + 3 = -x^2 + 2x + 2\\ \text{(d) right 1, reflect, up 3}\end{array}}$$

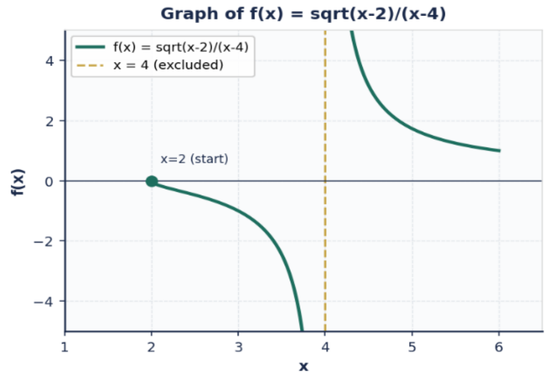
## Problem 21 :

**Answer: (a)** $(f \circ g)(x) = f(g(x)) = \sqrt{(x^2-2)+2} = \sqrt{x^2} = |x|$.
**The square root is defined when** $x^2 - 2 \geq -2 \Rightarrow x^2 \geq 0$**, always true.**
**Domain:** $\boxed{\mathbb{R}}$

**(b)** $(g \circ f)(x) = g(f(x)) = (\sqrt{x+2})^2 - 2 = x + 2 - 2 = x$.

**Domain of $f$ is $x \geq -2$ (since $x + 2 \geq 0$).**
**Domain: $\boxed{[-2, +\infty[}$**
**(c) As computed in (b), $(g \circ f)(x) = x$ for all $x \geq -2$. ✓**
**(d) For $f$ and $g$ to be inverses, we need both $f(g(x)) = x$ and $g(f(x)) = x$ on the respective domains.**
**We have $g(f(x)) = x$ on $[-2, +\infty[$, but $f(g(x)) = \mid x \mid$, which equals $x$ only for $x \geq 0$.**
**Thus they are not inverses of each other.**
**(However, if we restrict the domain of $g$ to $[0, +\infty[$, then $f$ and $g$ become inverses.)**
**Answer: $\boxed{\text{No}}$**

## Problem 22 :



**Answer:**

**(a)** Domain: $x - 2 \geq 0$ and $x - 4 \neq 0 \Rightarrow [2, \infty) \setminus \{4\}$.

**(b)** As $x \to 4^-$: numerator $\to \sqrt{2} > 0$, denominator $\to 0^- \Rightarrow f \to -\infty$.
As $x \to 4^+$: numerator $\to \sqrt{2} > 0$, denominator $\to 0^+ \Rightarrow f \to +\infty$.
Vertical asymptote: $x = 4$.

**(c)** $f(2) = \frac{0}{-2} = 0$, $f(3) = \frac{1}{-1} = -1$.

**(d)** $f(x) > 0$ for $x > 4$ (both numerator and denominator positive).
$f(x) < 0$ for $2 < x < 4$ (numerator positive, denominator negative).
$f(x) = 0$ at $x = 2$.

## Problem 23 :

**Answer:**

**(a)** $f(1) = f(0) + 2 = 5$
$f(2) = f(1) + 2 = 7$
$f(3) = f(2) + 2 = 9$
$f(-1) = f(0) - 2 = 1$
$f(-2) = f(-1) - 2 = -1$

**(b)** $f(n) = 2n + 3$ for all integers $n$.

**(c)** Base case $n = 0$: $f(0) = 3 = 2(0) + 3$ ✓
Inductive step: Assume $f(k) = 2k + 3$. Then
$f(k+1) = f(k) + 2 = 2k + 3 + 2 = 2(k+1) + 3$. ✓
So true for all $n \geq 0$.

**(d)** Arithmetic sequence with common difference $d = 2$.

.

## Problem 24 :

**Answer:**

**(a)** $f(x) = (x-3)^2 + 2$: axis of symmetry $x = 3$, vertex $(3, 2)$.

**(b)** $g(x) = \frac{1}{x-2} + 5$: centre of symmetry at $(2, 5)$ (intersection of asymptotes).

**(c)** $h(x) = -(x+1)^3 + 4$: centre of symmetry at $(-1,4)$ (translated $-x^3$).

$$\boxed{\text{(a) Axis } x = 3 \text{(b) Centre } (2,5) \text{(c) Centre } (-1,4)}$$

## Problem 25 :

**Answer:**

**(a)** $f(x) = \frac{(x-1)(x+1)}{x-1} = x + 1$ for $x \neq 1$.
Domain of $f$: $\mathbb{R} \setminus \{1\}$. Domain of $g$: $\mathbb{R}$.
They differ at $x = 1$ (f is undefined there).

**(b)** $f(x) = \sqrt{x^2} = \mid x \mid$, $g(x) = x$.
They differ for $x < 0$ (where $\mid x \mid = -x \neq x$).

**(c)** $f(x) = \frac{(x+2)^2}{x+2} = x + 2$ for $x \neq -2$.
They differ at $x = -2$ (f is undefined there).

$$\boxed{\text{(a) Differ at } x = 1 \text{(b) Differ for } x < 0 \text{(c) Differ at } x = -2}$$

.

## Problem 26 :

**Answer: (a)** $x^2 - x - 2 = (x-2)(x+1)$
Zeros: $x = -1, 2$.

**(b)** Sign table for $f'(x)$:

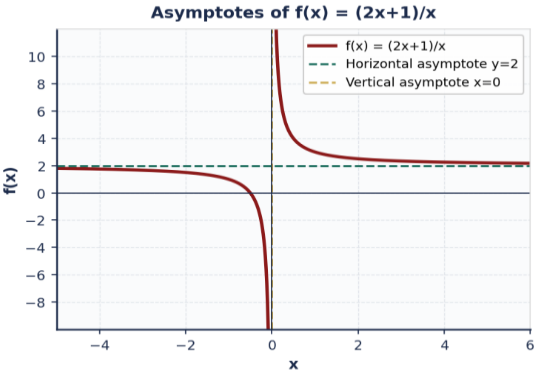
| Interval | $x+1$ | $x-2$ | $f'(x)$ |
|---|---|---|---|
| $(-\infty, -1)$ | $-$ | $-$ | $+$ |
| $(-1, 2)$ | $+$ | $-$ | $-$ |
| $(2, \infty)$ | $+$ | $+$ | $+$ |

**(c)** Variation table:

| $x$ | $-\infty$ | | $-1$ | | $2$ | | $+\infty$ |
|---|---|---|---|---|---|---|---|
| $f'(x)$ | | $+$ | $0$ | $-$ | $0$ | $+$ | |
| $f(x)$ | $-\infty$ | $\nearrow$ | $f(-1)$ | $\searrow$ | $f(2)$ | $\nearrow$ | $+\infty$ |

**(d)** $f(-1) = -\frac{1}{3} - \frac{1}{2} + 2 + 1 = \frac{13}{6} \approx 2.17$ (local max)
$f(2) = \frac{8}{3} - 2 - 4 + 1 = -\frac{7}{3} \approx -2.33$ (local min)

## Problem 27 :



**Answer:**

**(a)** $f(x) = \frac{2x+1}{x} = 2 + \frac{1}{x}$.
$A = 2$, $B = 1$.

**(b)** Horizontal asymptote: $y = 2$ (from the graph).
Vertical asymptote: $x = 0$ (from the graph).
Confirmed by formula: as $x \to \infty$, $f(x) \to 2$; as $x \to 0$, $f(x) \to \pm\infty$.

**(c)** $1/x$ is decreasing on each branch, so $f$ is decreasing on $(-\infty, 0)$ and on $(0, \infty)$.

**(d)** $2 + \frac{1}{x} > 2 \Rightarrow \frac{1}{x} > 0 \Rightarrow x > 0$.
Solution: $(0, \infty)$. Graphically, the curve is above $y = 2$ for $x > 0$.

$$\boxed{\begin{array}{c}\text{(a) } f(x) = 2 + \dfrac{1}{x} \\ \text{(b) } y = 2,\ x = 0 \\ \text{(c) Decreasing on both branches} \\ \text{(d) } (0, \infty)\end{array}}$$

## Problem 28 :

**Answer:**

**(a)** $f(x) = -3(x-1)^2 + 2$.
Reference: $-(x-1)^2$ is increasing on $(-\infty, 1]$.
So $f$ is **increasing** on $(-\infty, 1]$.

**(b)** $g(x) = 2 \cdot \frac{1}{x-3}$.
Reference: $\frac{1}{x-3}$ is decreasing on $(3, \infty)$.
Scaled by 2: still **decreasing**.

**(c)** $h(x) = \sqrt{2x+4} = \sqrt{2} \cdot \sqrt{x+2}$.
Reference: $\sqrt{x+2}$ is increasing on $[-2, \infty)$.
So $h$ is **increasing** on $[-2, \infty)$.

$$\boxed{\text{(a) Increasing(b) Decreasing(c) Increasing}}$$

## Problem 29 :

**Answer:**

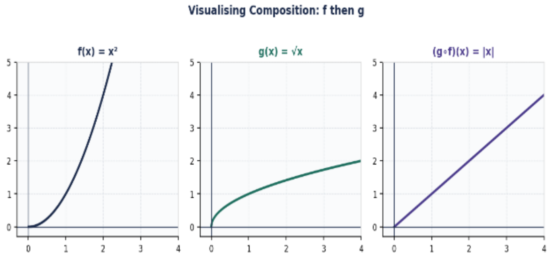
**(a)** The graph is symmetric about the line $x = 2$.

**(b)** Example: $f(x) = (x-2)^2 + 1$.

**(c)** $f(2-x) = f(2+x)$:
Set $x = 2$: $f(0) = f(4) \Rightarrow f(4) = 3$.
Set $x = 1$: $f(1) = f(3) \Rightarrow f(3) = 5$.

**(d)** If $f$ is odd, then $f(-x) = -f(x)$.
For a function symmetric about $x = 2$ and odd, we would need very specific behaviour.
Generally, it cannot be both unless $f \equiv 0$, which contradicts $f(1) = 5$. So **no**.

## Problem 30 :



**Answer:**

**(a)** $(g \circ f)(x) = g(f(x)) = \sqrt{x^2} = |x|$.
For $x \geq 0$, $|x| = x$. For $x < 0$, $|x| = -x$.

**(b)** They differ for $x < 0$. The composition loses the sign information.

**(c)** $(f \circ g)(x) = f(g(x)) = (\sqrt{x})^2 = x$ for $x \geq 0$.

**(d)** The order matters: $f$ then $g$ loses the sign; $g$ then $f$ recovers $x$ exactly on $[0, \infty)$.
Composition is not commutative.

## Problem 31 :

**Answer:**

**(a)** $k = 0$: $f(x) = 3 > 0$ for all $x$. Works.

**(b)** For $k > 0$, the quadratic opens upward. It is strictly positive for all $x$ if its discriminant is negative: $\Delta < 0$.
$\Delta = (2k)^2 - 4k(k+3) = 4k^2 - 4k^2 - 12k = -12k$.
$\Delta < 0 \Rightarrow -12k < 0 \Rightarrow k > 0$.

**(c)** All $k > 0$ work. Combined with $k = 0$:

$$\boxed{k \geq 0}$$

## Problem 32 :

**Answer:**

**(a)** $f(x) = \begin{cases} x^2 - 4 & \text{if } x \leq -2 \text{ or } x \geq 2 \\ -(x^2-4) = 4 - x^2 & \text{if } -2 < x < 2 \end{cases}$

**(b)** Zeros: $x = \pm 2$.
$f(x) = x^2 - 4$ on $(-\infty, -2] \cup [2, \infty)$.
$f(x) = 4 - x^2$ on $(-2, 2)$.

**(c)** Graph: parabola opening up for $|x| \geq 2$, inverted parabola for $|x| < 2$.
Zeros at $x = \pm 2$. Local minima at $x = \pm 2$ (value 0).
Local maximum at $x = 0$ (value 4).

**(d)** $f(-x) = |(-x)^2 - 4| = |x^2 - 4| = f(x) \to$ **Even**.

## Problem 33 :

**Answer:**

**(a)** Quadratic, opens downward. Domain: $[0, t_{\text{ground}}]$, where $t_{\text{ground}}$ is the positive root of $H(t) = 0$.

**(b)** $H(t) = -5(t^2 - 3t) + 1.5 = -5[(t - 1.5)^2 - 2.25] + 1.5 = -5(t - 1.5)^2 + 11.25 + 1.5 = -5(t - 1.5)^2 + 12.75$.
Maximum height: 12.75 m at $t = 1.5$ s.

**(c)** $-5t^2 + 15t + 1.5 = 0 \Rightarrow$ multiply by 2 to avoid decimals: $-10t^2 + 30t + 3 = 0$.
$\Delta = 30^2 - 4(-10)(3) = 900 + 120 = 1020$.
$t = \frac{-30 \pm \sqrt{1020}}{-20} = \frac{30 \pm \sqrt{1020}}{20} = \frac{30 \pm 2\sqrt{255}}{20} = \frac{15 \pm \sqrt{255}}{10}$.
Positive root: $t = \frac{15+\sqrt{255}}{10} \approx 3.097$ s.

**(d)** Range: $[0, 12.75]$ m.

## Problem 34 :

**Answer:**

Let $m = -\frac{b}{2a}$. We need to show $f(m + t) = f(m - t)$ for any real $t$.

$$\begin{aligned} f(m+t) &= a(m+t)^2 + b(m+t) + c \\ &= a(m^2 + 2mt + t^2) + b(m+t) \\ &\quad + c \\ f(m-t) &= a(m-t)^2 + b(m-t) + c \\ &= a(m^2 - 2mt + t^2) + b(m-t) \\ &\quad + c \end{aligned}$$

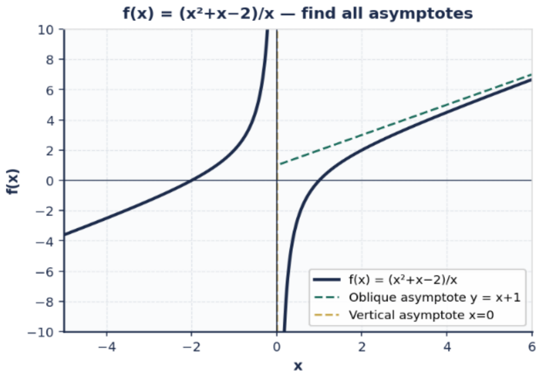
Difference:

$$\begin{aligned} f(m+t) - f(m-t) &= a(4mt) + b(2t) \\ &= 2t(2am + b) \end{aligned}$$

Since $m = -\frac{b}{2a}$, we have $2am + b = 2a\left(-\frac{b}{2a}\right) + b = -b + b = 0$.
So $f(m + t) = f(m - t)$ for all $t$. ✓

.

## Problem 35 :



**Answer:**

**(a)** $x^2 + x - 2 = x(x + 1) - 2$, so quotient $x + 1$, remainder $-2$.

**(b)** $f(x) = x + 1 - \frac{2}{x}$.

**(c)** Vertical asymptote: $x = 0$. Oblique asymptote: $y = x + 1$.

**(d)** $f(x) - (x + 1) = -\frac{2}{x}$.
For $x > 0$, negative → curve **below** the asymptote.
For $x < 0$, positive → curve **above** the asymptote.

# Problems 36–50 — Difficult & Real-World Applications

## Problem 36 :

**Answer:**

**(a)** $f(0 + 0) = f(0) + f(0) \Rightarrow f(0) = 2f(0) \Rightarrow f(0) = 0$.

**(b)** $0 = f(0) = f(x + (-x)) = f(x) + f(-x) \Rightarrow f(-x) = -f(x)$.

**(c)** By induction: $f(nx) = f((n - 1)x + x) = f((n - 1)x) + f(x) = (n - 1)f(x) + f(x) = nf(x)$. ✓

**(d)** $f(n) = nf(1) = nk$.
For $p/q$: $f(p) = f\left(q \cdot \frac{p}{q}\right) = qf\left(\frac{p}{q}\right) \Rightarrow kp = qf\left(\frac{p}{q}\right) \Rightarrow f\left(\frac{p}{q}\right) = \frac{kp}{q} = k\left(\frac{p}{q}\right)$. ✓

## Problem 37 :

**Answer:**

**(a)** $f(x_2) - f(x_1) = x_2^2 - x_1^2 = (x_2 - x_1)(x_2 + x_1)$.

**(b)** If $0 \le x_1 < x_2$, then $x_2 - x_1 > 0$ and $x_2 + x_1 > 0$.
So their product is positive: $f(x_2) - f(x_1) > 0$.

**(c)** Therefore $f(x_1) < f(x_2)$. So $f$ is strictly increasing on $[0, \infty)$.

**(d)** $f$ is **not** increasing on all $\mathbb{R}$.
Counterexample: $-3 < -1$ but $f(-3) = 9 > 1 = f(-1)$. ✓

## Problem 38 :

**Answer:**

**(a)** $x_1 < x_2$: $f(x_1) < f(x_2)$ and $g(x_1) < g(x_2)$.
Adding: $(f + g)(x_1) < (f + g)(x_2)$. ✓

**(b)** Not necessarily. Counterexample: $f(x) = x$, $g(x) = x - 10$ on $[0, 5]$.
Both are increasing, but $f(1)g(1) = 1(-9) = -9$, $f(2)g(2) = 2(-8) = -16$. Product decreases.

**(c)** $f - g = f + (-g)$. Since $g$ is decreasing, $-g$ is increasing.
So $f - g$ is increasing.

**(d)** For $x_1 < x_2, f, g \geq 0$:
$f(x_2)g(x_2) - f(x_1)g(x_1) = f(x_2)(g(x_2) - g(x_1)) + g(x_1)(f(x_2) - f(x_1)) \geq 0$.
Thus $f \cdot g$ is increasing.

## Problem 39 :

**Answer:**

**(a)** $C(n) = \frac{1}{10}(n^2 - 60n) + 100 = \frac{1}{10}[(n-30)^2 - 900] + 100 = \frac{1}{10}(n-30)^2 - 90 + 100 = \frac{1}{10}(n-30)^2 + 10$.
Vertex: $(30, 10)$.

**(b)** Minimum cost at $n = 30$ deliveries: $C(30) = 10$ thousand DA = 10 000 DA.

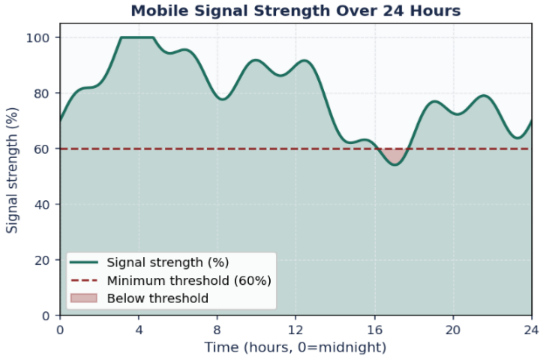
**(c)** Revenue: $R(n) = 4n$ (thousands DA, since 4000 DA = 4 thousand DA).

$P(n) = R(n) - C(n) = 4n - \left(\frac{n^2}{10} - 6n + 100\right) = -\frac{n^2}{10} + 10n - 100$.

**(d)** Complete the square for $P$:
$P(n) = -\frac{1}{10}(n^2 - 100n) - 100 = -\frac{1}{10}[(n-50)^2 - 2500] - 100 = -\frac{1}{10}(n-50)^2 + 250 - 100 = -\frac{1}{10}(n-50)^2 + 150$.
Maximum at $n = 50$ deliveries.
$P(50) = 150$ thousand DA = 150 000 DA.

.

## Problem 40 :



**Answer:**

**(a)** From the graph, signal is below 60% approximately during $t \approx 5$–8 h and $t \approx 18$–21 h.

**(b)** $S(t) = 80 + 15\sin\left(\frac{2\pi t}{24}\right)$
Maximum: $80 + 15 = 95\%$. Minimum: $80 - 15 = 65\%$.

**(c)** $80 + 15\sin\left(\frac{2\pi t}{24}\right) \geq 60$ always, since the minimum is 65%.
So satisfied for all $t$.

**(d)** The simplified model gives 100% uptime (since signal never drops below 60%).

The real graph dips below 60%, so the simple model is too optimistic.

## .Problem 41 :

**Answer:**

**(a)** $ax + b = \frac{a}{c}(cx + d) + \left(b - \frac{ad}{c}\right)$
So $f(x) = \frac{a}{c} + \frac{bc - ad}{c(cx+d)}$.
$A = \frac{a}{c}$, $B = \frac{bc-ad}{c}$.

**(b)** Horizontal asymptote: $y = \frac{a}{c}$. Vertical asymptote: $x = -\frac{d}{c}$.

**(c)** $f'(x) = \frac{ad - bc}{(cx+d)^2}$ (from the form $A + B/(cx + d)$).
Sign of $f'$ = sign of $ad - bc$.
If $ad - bc > 0$: increasing. If $<0$: decreasing.

**(d)** Centre of symmetry: intersection of asymptotes $\left(-\frac{d}{c}, \frac{a}{c}\right)$.
For $f(x) = \frac{3x-2}{x+1}$: $a = 3, b = -2, c = 1, d = 1$.
Asymptotes: $x = -1$, $y = 3$. Centre: $(-1,3)$. ✓

.

## Problem 42 :

**Answer:**

**(a)** $f(x) - 2 = x + \frac{1}{x} - 2 = \frac{x^2 - 2x + 1}{x} = \frac{(x-1)^2}{x} \geq 0$ since $x > 0$.
So $f(x) \geq 2$. Equality when $x - 1 = 0 \Rightarrow x = 1$.

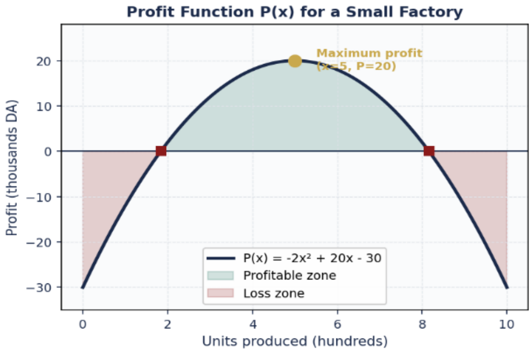
**(b)** For $0 < x_1 < x_2 \leq 1$:
$f(x_2) - f(x_1) = (x_2 - x_1) + \left(\frac{1}{x_2} - \frac{1}{x_1}\right) = (x_2 - x_1) + \frac{x_1 - x_2}{x_1 x_2} = (x_2 - x_1)\left(1 - \frac{1}{x_1 x_2}\right)$.
Since $x_1 x_2 < 1$, $1 - \frac{1}{x_1 x_2} < 0$, so $f(x_2) - f(x_1) < 0$ → decreasing.
Similarly, for $1 \leq x_1 < x_2$, $x_1 x_2 > 1$ → increasing.

**(c)** Range: $[2, \infty)$.

**(d)** $x + \frac{1}{x} \leq \frac{5}{2} \Rightarrow 2x^2 + 2 \leq 5x \Rightarrow 2x^2 - 5x + 2 \leq 0 \Rightarrow (2x-1)(x-2) \leq 0$.
Roots: $x = \frac{1}{2}, 2$. Parabola opens upward → between roots.

$$\boxed{\left[\frac{1}{2}, 2\right]}$$

## Problem 43 :



**Answer:**

**(a)** $-2x^2 + 20x - 30 = 0 \Rightarrow x^2 - 10x + 15 = 0$.
$\Delta = 100 - 60 = 40$.
$x = \frac{10 \pm \sqrt{40}}{2} = 5 \pm \sqrt{10}$.
$x_1 \approx 1.84$, $x_2 \approx 8.16$.

**(b)** Vertex: $x = 5$, $P(5) = -50 + 100 - 30 = 20$.
Max profit: 20 000 DA at 500 units.

**(c)** Capacity: 900 units $\Rightarrow x \leq 9$.
$x = 5$ is within capacity, so optimal still $x = 5$.

**(d)** $P_{\text{new}}(x) = -2x^2 + 20x - 40 = 0 \Rightarrow x^2 - 10x + 20 = 0$.
$\Delta = 100 - 80 = 20 > 0$.
So it still breaks even (two break-even points: $x = 5 \pm \sqrt{5}$).

## Problem 44 :

**Answer:**

**(a)** $\sqrt{x} = 1 - x$: need $x \geq 0$ and $1 - x \geq 0 \Rightarrow 0 \leq x \leq 1$.
Square: $x = (1-x)^2 = 1 - 2x + x^2 \Rightarrow x^2 - 3x + 1 = 0$
$x = \frac{3 \pm \sqrt{5}}{2}$. Only $x = \frac{3-\sqrt{5}}{2} \approx 0.382$ is in $[0,1]$.

**(b)** At $x = 0$: $f = 0$, $g = 1 \Rightarrow g > f$.
At $x = 1$: $f = 1$, $g = 0 \Rightarrow f > g$.
So $f > g$ on $(\frac{3-\sqrt{5}}{2}, 1]$.

**(c)** $(f \circ g)(x) = \sqrt{1-x}$, domain $x \leq 1$.
$(g \circ f)(x) = 1 - \sqrt{x}$, domain $x \geq 0$.

**(d)** $\sqrt{1-x} = 1 - \sqrt{x}$.
Square: $1 - x = 1 - 2\sqrt{x} + x \Rightarrow 2\sqrt{x} = 2x \Rightarrow \sqrt{x} = x \Rightarrow x = x^2 \Rightarrow x = 0$ or $x = 1$.
Check: $x = 0$: $f(g(0)) = \sqrt{1} = 1$, $g(f(0)) = 1 - 0 = 1$ ✓
$x = 1$: $f(g(1)) = \sqrt{0} = 0$, $g(f(1)) = 1 - 1 = 0$ ✓

$$\boxed{x = 0,\ x = 1}$$

## Problem 45 :

**Answer:**

**(a)** The graph of $f$ lies on or below the chord connecting any two points.

**(b)** Let $x_1, x_2 \in \mathbb{R}$, $t \in [0,1]$.

$$f(tx_1 + (1-t)x_2) = (tx_1 + (1-t)x_2)^2$$

RHS = $tx_1^2 + (1-t)x_2^2$.
Difference: RHS - LHS = $t(1-t)(x_1 - x_2)^2 \geq 0$.
So $f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2)$ ✓

**(c)** $f(tx_1 + (1-t)x_2) = m(tx_1 + (1-t)x_2) + b = t(mx_1 + b) + (1-t)(mx_2 + b) = tf(x_1) + (1-t)f(x_2)$.
Equality holds → both convex and concave.

**(d)** Use convexity of $x^2$ with $t = \frac{1}{2}$, $x_1 = a$, $x_2 = b$:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2}f(a) + \frac{1}{2}f(b)$$

$\left(\frac{a+b}{2}\right)^2 \leq \frac{a^2+b^2}{2}$. ✓

## Problem 46 :

**Answer:**

**(a)** $R(p) = p \cdot D(p) = p\left(500 - \frac{p}{10}\right) = 500p - \frac{p^2}{10}$.

**(b)** $R'(p) = 500 - \frac{p}{5} = 0 \Rightarrow p = 2500$.
$R(2500) = 500(2500) - \frac{2500^2}{10} = 1\,250\,000 - 625\,000 = 625\,000$ DA.

**(c)** $P(p) = R(p) - 1500D(p) = 500p - \frac{p^2}{10} - 1500\left(500 - \frac{p}{10}\right)$
$= 500p - \frac{p^2}{10} - 750\,000 + 150p = 650p - \frac{p^2}{10} - 750\,000$.

**(d)** $P'(p) = 650 - \frac{p}{5} = 0 \Rightarrow p = 3250$.
The profit-maximising price (3250 DA) is higher than the revenue-maximising price (2500 DA). Economically, this makes sense: the shop must cover the cost of goods, so a higher margin per sale is needed, justifying a lower sales volume.

## Problem 47 :

**Answer:**

**(a)** $\lfloor 0.9 \rfloor = 0$, $\lfloor 3 \rfloor = 3$, $\lfloor -0.1 \rfloor = -1$, $\lfloor -\pi \rfloor = -4$.

**(b)** Domain: $\mathbb{R}$. Range: $\mathbb{Z}$ (all integers).

**(c)** $f$ is non-decreasing (weakly increasing). It is neither even nor odd (e.g., $\lfloor -0.3 \rfloor = -1$, but $\lfloor 0.3 \rfloor = 0$).

**(d)** $\lfloor 2x - 1 \rfloor = 3 \Rightarrow 3 \leq 2x - 1 < 4 \Rightarrow 4 \leq 2x < 5 \Rightarrow 2 \leq x < \frac{5}{2}$.

$$\boxed{[2, \frac{5}{2})}$$

## Problem 48 :

**Answer:**

**(a)** $C(0) = \frac{4(0)}{0^2+1} = 0$ mg/L.
Interpretation: immediately after taking the tablet, the concentration is zero (no drug has entered the bloodstream yet).

$C(1) = \frac{4(1)}{1^2+1} = \frac{4}{2} = 2$ mg/L.

$C(3) = \frac{4(3)}{3^2+1} = \frac{12}{10} = 1.2$ mg/L.

**(b)** $C(t) = \frac{4t}{t^2+1} = \frac{4}{t+\frac{1}{t}}$ (divide numerator and denominator by $t$, valid for $t > 0$).

By the AM-GM inequality: for $t > 0$, $t + \frac{1}{t} \geq 2\sqrt{t \cdot \frac{1}{t}} = 2$.

Therefore, $C(t) = \frac{4}{t+\frac{1}{t}} \leq \frac{4}{2} = 2$.

Thus $C(t) \leq 2$ for all $t > 0$. ✓

**(c)** $C(1) = \frac{4(1)}{1^2+1} = \frac{4}{2} = 2$.

In the AM-GM inequality, equality occurs when $t = \frac{1}{t}$, i.e., $t = 1$.
So the bound is tight at $t = 1$, confirming that $C(1) = 2$ is the maximum concentration.

**(d)** We need $C(t) \geq 1$:

$$\frac{4t}{t^2+1} \geq 1$$

Multiply both sides by $t^2 + 1 > 0$:

$$4t \geq t^2 + 1$$
$$0 \geq t^2 - 4t + 1 \Rightarrow t^2 - 4t + 1 \leq 0$$

Solve $t^2 - 4t + 1 = 0$:

$$\Delta = (-4)^2 - 4(1)(1) = 16 - 4 = 12$$
$$t = \frac{4 \pm \sqrt{12}}{2} = \frac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}$$

The quadratic $t^2 - 4t + 1$ opens upward, so it is negative between the roots:

$$2 - \sqrt{3} \leq t \leq 2 + \sqrt{3}$$

Approximately: $0.27 \leq t \leq 3.73$ hours.

## Problem 49 :

**Answer:**

**(a)** $a_0 = 0$

$$a_1 = f(a_0) = 0^2 - 2 = -2$$
$$a_2 = f(a_1) = (-2)^2 - 2 = 4 - 2 = 2$$
$$a_3 = f(a_2) = 2^2 - 2 = 4 - 2 = 2$$
$$a_4 = f(a_3) = 2^2 - 2 = 2$$

Notice: once the sequence reaches 2, it stays there.

**(b)** Fixed points satisfy $f(x) = x$:
$x^2 - 2 = x \Rightarrow x^2 - x - 2 = 0 \Rightarrow (x-2)(x+1) = 0$
Fixed points: $x = 2$ and $x = -1$.

**(c)** $a_{n+1} - a_n = (a_n^2 - 2) - a_n = a_n^2 - a_n - 2$
For $a_n > 2$: $a_n^2 - a_n - 2 = (a_n - 2)(a_n + 1) > 0$ (since both factors are positive).
So $a_{n+1} - a_n > 0 \Rightarrow$ the sequence is increasing.
This suggests that if the sequence ever exceeds 2, it will grow without bound (diverge to $+\infty$).

**(d)** Starting from $a_0 = 0$:

$$a_0 = 0$$
$$a_1 = -2$$
$$a_2 = 2$$
$$a_3 = 2$$
$$a_4 = 2$$
$$\vdots$$

The sequence reaches the fixed point $x = 2$ after two steps and stays there.
So it **converges to 2**.

## Problem 50 :

**Answer:**

**(a)** The distance from $P(x, y)$ to the focus $F(0,2)$ is:

$$PF = \sqrt{(x-0)^2 + (y-2)^2} = \sqrt{x^2 + (y-2)^2}$$

The distance from $P$ to the directrix $y = -2$ is:

$$\text{distance} = |\, y - (-2) \,| = |\, y + 2 \,|$$

For points above the directrix (which is all points on the parabola), $y + 2 \geq 0$, so the distance is $y + 2$.

Set them equal:

$$\sqrt{x^2 + (y-2)^2} = y + 2$$

Square both sides:

$$x^2 + (y-2)^2 = (y+2)^2$$

Expand:

$$x^2 + y^2 - 4y + 4 = y^2 + 4y + 4$$

Simplify:

$$x^2 - 4y = 4y \Rightarrow x^2 = 8y$$

$$\boxed{x^2 = 8y}$$

**(b)** $y = \frac{x^2}{8}$. Domain: all real numbers $\mathbb{R}$.

$$\boxed{y = \frac{x^2}{8}, \text{ domain } \mathbb{R}}$$

**(c)** The vertex of the parabola is at $(0, 0)$.
At the rim, $x = \pm 60$:

$$y = \frac{60^2}{8} = \frac{3600}{8} = 450 \text{ cm}$$

The depth is the difference in y-values between the rim and the vertex:

$$\text{depth} = 450 - 0 = 450 \text{ cm} = 4.5 \text{ m}$$

$$\boxed{450 \text{ cm} = 4.5 \text{ m}}$$

**(d)** At $x = 40$ cm:

$$y = \frac{40^2}{8} = \frac{1600}{8} = 200 \text{ cm}$$

So the point is $P(40,200)$.

By the definition of the parabola, $PF$ equals the distance from $P$ to the directrix $y = -2$:

$$PF = 200 - (-2) = 202$$

The reflective property of a parabola states that any ray parallel to the axis (the y-axis) will reflect through the focus.
Since the incoming signal is vertical (parallel to the axis), it will reflect toward the focus $F(0,2)$.

$$\boxed{\text{The ray reflects through the focus } F(0,2)}$$

CHAPTER 3 — EQUATIONS, INEQUALITIES & LIMITS

# SOLUTIONS OF PROBLEMS

## Solutions to Problems 1–10 — Instant Recall

### Problem 1 :

**Answer:**

**(a)** $3x - 6 > 0$
$3x > 6 \Rightarrow x > 2$

$$\boxed{(2, \infty)}$$

**(b)** $-2x + 8 \geq 0$
$-2x \geq -8 \Rightarrow x \leq 4$ (divide by $-2$, reverse inequality)

$$\boxed{(-\infty, 4]}$$

**(c)** $5 - x < 0$
$-x < -5 \Rightarrow x > 5$

$$\boxed{(5, \infty)}$$

**(d)** $4x + 12 \leq 0$
$4x \leq -12 \Rightarrow x \leq -3$

$$\boxed{(-\infty, -3]}$$

### Problem 2 :

**Answer:**

**(a)** Roots: $x = 2$, $x = -5$.
Parabola opens upward. Positive outside the roots.

$$\boxed{(-\infty, -5) \cup (2, \infty)}$$

**(b)** Roots: $x = -1$, $x = 4$.
Parabola opens upward. Negative between the roots.

$$\boxed{[-1, 4]}$$

**(c)** Roots: $x = \frac{1}{2}$, $x = -3$.
Parabola opens upward. Negative between the roots.

$$\boxed{\left(-3, \frac{1}{2}\right)}$$

**(d)** Critical values: $x = 3$ (numerator zero), $x = -1$ (denominator zero).
Sign table:

| Interval | $x - 3$ | $x + 1$ | Quotient |
|---|---|---|---|
| $]-\infty, -1]$ | $-$ | $-$ | $+$ |
| $[-1, 3]$ | $-$ | $+$ | $-$ |
| $[3, +\infty[$ | $+$ | $+$ | $+$ |

We want $> 0$: $]-\infty, -1] \cup [3, +\infty[$.

$$\boxed{]-\infty, -1] \cup [3, +\infty[}$$

### Problem 3 :

**Answer:**

**(a)** Substitute $x = 3$: $2(9) - 5(3) + 1 = 18 - 15 + 1 = 4$.

$$\boxed{4}$$

**(b)** Substitute $x = -1$: $(-1)^3 + 2(-1) - 4 = -1 - 2 - 4 = -7$.

$$\boxed{-7}$$

**(c)** Substitute $x = 0$: $\sqrt{0 + 9} = \sqrt{9} = 3$.

$$\boxed{3}$$

**(d)** Substitute $x = 2$: $\frac{4-3}{2+1} = \frac{1}{3}$.

$$\boxed{\frac{1}{3}}$$

### Problem 4 :

**Answer:**

**(a)** Divide numerator and denominator by $x$:
$\frac{3+\frac{1}{x}}{1-\frac{2}{x}} \to \frac{3+0}{1-0} = 3$.

$$\boxed{3}$$

**(b)** Divide by $x^2$:
$\frac{5-\frac{2}{x^2}}{1+\frac{3}{x^2}} \to \frac{5-0}{1+0} = 5$.

$$\boxed{5}$$

**(c)** Divide by $x^2$:
$\frac{\frac{1}{x}+\frac{4}{x^2}}{1-\frac{1}{x^2}} \to \frac{0+0}{1-0} = 0$.

$$\boxed{0}$$

**(d)** Divide by $x^3$:
$\frac{2}{1+\frac{5}{x^3}} \to \frac{2}{1+0} = 2$.

$$\boxed{2}$$

### Problem 5 :

**Answer:**

**(a)** Denominator zero: $x - 4 = 0 \Rightarrow x = 4$.
Numerator not zero at $x = 4$.

$$\boxed{x = 4}$$

**(b)** Denominator zero: $x = 1$ and $x = -3$.
Numerator not zero at either.

$$\boxed{x = 1,\ x = -3}$$

**(c)** Denominator zero: $x^2 - 9 = 0 \Rightarrow x = \pm 3$. Numerator not zero at $x = \pm 3$ (it is 9 at both).

$$\boxed{x = 3,\ x = -3}$$

**(d)** Denominator $x^2 + 1$ is never zero for real $x$.

$$\boxed{\text{No vertical asymptotes}}$$

## Problem 6 :

### Answer:

**(a)** Factor: $(x-3)(x+2) > 0$. Roots: $3, -2$. Parabola opens upward. Positive outside the roots.

$$\boxed{(-\infty, -2) \cup (3, \infty)}$$

**(b)** Factor: $(x+4)(x-2) \leq 0$. Roots: $-4, 2$. Parabola opens upward. Negative between the roots.

$$\boxed{[-4, 2]}$$

**(c)** Factor: $-(x-5)(x+2) \geq 0 \Rightarrow (x-5)(x+2) \leq 0$. Roots: $5, -2$. Parabola opens upward (after removing the negative). Negative between roots.

$$\boxed{[-2, 5]}$$

**(d)** $2x^2 - 8 < 0 \Rightarrow 2(x^2 - 4) < 0 \Rightarrow x^2 - 4 < 0 \Rightarrow (x-2)(x+2) < 0$. Roots: $2, -2$. Negative between roots.

$$\boxed{(-2, 2)}$$

## Problem 7 :

### Answer:

**(a)** $\frac{x^2-4}{x-2} = \frac{(x-2)(x+2)}{x-2} = x + 2$ for $x \neq 2$.
$\lim\limits_{x \to 2}(x+2) = 4$.

$$\boxed{4}$$

**(b)** $\frac{x^2+5x+6}{x+3} = \frac{(x+3)(x+2)}{x+3} = x + 2$ for $x \neq -3$.
$\lim\limits_{x \to -3}(x+2) = -1$.

$$\boxed{-1}$$

**(c)** $x^3 - 1 = (x-1)(x^2 + x + 1)$.
$\frac{x^3-1}{x-1} = x^2 + x + 1$ for $x \neq 1$.
$\lim\limits_{x \to 1}(x^2 + x + 1) = 1 + 1 + 1 = 3$.

$$\boxed{3}$$

**(d)** $\frac{x^2+3x}{x} = \frac{x(x+3)}{x} = x + 3$ for $x \neq 0$.
$\lim\limits_{x \to 0}(x+3) = 3$.

$$\boxed{3}$$

## Problem 8 :

### Answer:

**(a)** Degree numerator = degree denominator = 1. Ratio of leading coefficients: $\frac{4}{2} = 2$.

$$\boxed{y = 2}$$

**(b)** Degree numerator = degree denominator = 2. Ratio of leading coefficients: $\frac{1}{3}$.

$$\boxed{y = \frac{1}{3}}$$

**(c)** Degree numerator (1) < degree denominator (2).

$$\boxed{y = 0}$$

**(d)** Degree numerator = degree denominator = 3. Ratio of leading coefficients: $\frac{1}{1} = 1$.

$$\boxed{y = 1}$$

## Problem 9 :

### Answer:

**(a)** $\frac{1}{x-3} > 0$ means denominator positive: $x - 3 > 0 \Rightarrow x > 3$.

$$\boxed{(3, \infty)}$$

**(b)** Critical values: $x = -2$ (numerator zero), $x = 1$ (denominator zero).
Sign table:

| Interval | $x+2$ | $x-1$ | Quotient |
|---|---|---|---|
| $]-\infty, -2]$ | $-$ | $-$ | $+$ |
| $[-2, 1]$ | $+$ | $-$ | $-$ |
| $[1, +\infty[$ | $+$ | $+$ | $+$ |

We want $<0$: $[-2, 1]$.

$$\boxed{[-2, 1]}$$

**(c)** Critical values: $x = 4$ (numerator zero), $x = -2$ (denominator zero, exclude).
Sign table:

| Interval | $x-4$ | $x+2$ | Quotient |
|---|---|---|---|
| $]-\infty, -2]$ | $-$ | $-$ | $+$ |
| $[-2, 4]$ | $-$ | $+$ | $-$ |
| $54, = \infty[$ | $+$ | $+$ | $+$ |

We want $\geq 0$: $]-\infty, -2] \cup [4, +\infty[$.

$$\boxed{]-\infty, -2] \cup [4, +\infty[}$$

**(d)** $\frac{2}{x+5} \leq 1$. Bring to one side: $\frac{2}{x+5} - 1 \leq 0 \Rightarrow \frac{2-(x+5)}{x+5} \leq 0 \Rightarrow \frac{-x-3}{x+5} \leq 0 \Rightarrow \frac{x+3}{x+5} \geq 0$.
Critical values: $x = -3$ (numerator zero), $x =$

$-5$ (denominator zero, exclude).
Sign table:

| Interval | $x+3$ | $x+5$ | Quotient |
|---|---|---|---|
| $]-\infty,-5]$ | $-$ | $-$ | $+$ |
| $[-5,-3]$ | $-$ | $+$ | $-$ |
| $]-3,+\infty[$ | $+$ | $+$ | $+$ |

We want $\geq 0$: $]-\infty,-5] \cup [-3,+\infty[$.

$$\boxed{]-\infty,-5] \cup [-3,+\infty[}$$

## Problem 10 :

**Answer:**

**(a)** $\frac{x^2+1}{x} = x + \frac{1}{x}$.
Oblique asymptote: $y = x$.

$$\boxed{y = x}$$

**(b)** Divide $x^2 - 3x + 1$ by $x - 2$:
$(x-2)(x-1) = x^2 - 3x + 2$.
$x^2 - 3x + 1 = (x-2)(x-1) - 1$.
So $g(x) = x - 1 - \frac{1}{x-2}$.
Oblique asymptote: $y = x - 1$.

$$\boxed{y = x - 1}$$

**(c)** Divide $2x^2 + x - 3$ by $x + 1$:
$(x+1)(2x-1) = 2x^2 + x - 1$.
$2x^2 + x - 3 = (x+1)(2x-1) - 2$.
So $h(x) = 2x - 1 - \frac{2}{x+1}$.
Oblique asymptote: $y = 2x - 1$.

$$\boxed{y = 2x - 1}$$

# Solutions to Problems 11–35 — Medium

## Problem 11 :

**Answer:**

**(a)** First inequality: $x^2 - x - 6 < 0$
Factor: $(x-3)(x+2) < 0$.
Roots: $x = 3$, $x = -2$. Parabola opens upward, so negative between the roots.
Solution: $(-2,3)$.

Second inequality: $|x - 1| \leq 3$
$-3 \leq x - 1 \leq 3 \Rightarrow -2 \leq x \leq 4$.
Solution: $[-2,4]$.

**(b)** Intersection: $(-2,3) \cap [-2,4] = (-2,3)$.

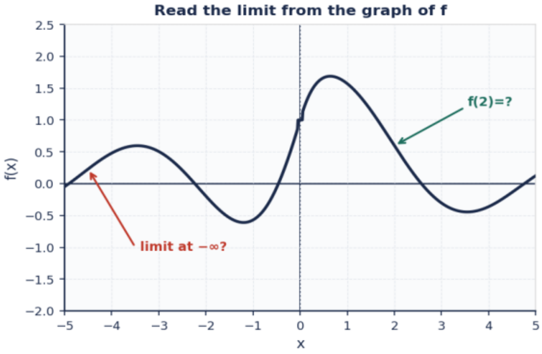
$$\boxed{(-2,3)}$$

**(c)** Verify:

- $x = 0$: satisfies both ✓
- $x = 3$: first inequality gives $0 < 0$ false (excluded) ✓
- $x = -2$: first inequality gives $0 < 0$ false (excluded) ✓

.

## Problem 12 :



**Answer:**

**(a)** From the graph, as $x \to 0$, $f(x)$ appears to approach $1$.
$f(0)$ may or may not exist (cannot be determined from the graph alone).

$$\boxed{\lim_{x\to 0} f(x) \approx 1}$$

**(b)** As $x \to -\infty$, $f(x) \to 0$. As $x \to +\infty$, $f(x) \to 0$.
Horizontal asymptote: $y = 0$.

$$\boxed{y = 0}$$

**(c)** A removable discontinuity occurs where the graph has a hole (open circle).

$$\boxed{\text{Hole at the point where the curve is broken}}$$

**(d)** Left-hand and right-hand limits at $x = 0$ both appear to approach $1$, so the two-sided limit exists.

## Problem 13 :

**Answer:**

**(a)** $|x^2 - 4| < 5 \Rightarrow -5 < x^2 - 4 < 5 \Rightarrow -1 < x^2 < 9$.
$x^2 > -1$ is always true.
$x^2 < 9 \Rightarrow |x| < 3 \Rightarrow (-3,3)$.

$$\boxed{(-3,3)}$$

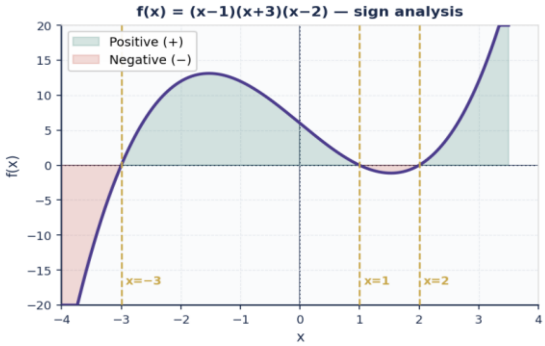
**(b)** $|x^2 - 4| > 5 \Rightarrow x^2 - 4 > 5$ OR $x^2 - 4 < -5$.
First: $x^2 > 9 \Rightarrow |x| > 3 \Rightarrow (-\infty,-3) \cup (3,\infty)$.
Second: $x^2 < -1$ (no real solution).

$$\boxed{(-\infty,-3) \cup (3,\infty)}$$

**(c)** $(x+1)(x-3) \leq 0 \Rightarrow [-1,3]$.
Intersect with $(-3,3)$: $[-1,3)$.

$$\boxed{[-1,3]}$$

## Problem 14 :



### Answer:

**(a)** Zeros: $x = -3$, $x = 1$, $x = 2$.

$$\boxed{-3,\ 1,\ 2}$$

**(b)** $f(x) > 0$ where the curve is above the x-axis:

$$\boxed{[-3,1] \cup [2,+\infty[}$$

**(c)** $f(x) \leq 0$ where the curve is on or below the x-axis:

$$\boxed{]-\infty,-3] \cup [1,2]}$$

**(d)** Sign table:

| Interval | $x+3$ | $x-1$ | $x-2$ | Product |
|---|---|---|---|---|
| $]-\infty,-3]$ | $-$ | $-$ | $-$ | $-$ |
| $[-3,1]$ | $+$ | $-$ | $-$ | $+$ |
| $[1,2]$ | $+$ | $+$ | $-$ | $-$ |
| $[2,+\infty[$ | $+$ | $+$ | $+$ | $+$ |

Positive on $[-3,1] \cup [2,+\infty[$. Confirmed ✓

## Problem 15 :

### Answer:

**(a)** Multiply by conjugate:

$\frac{\sqrt{x}-2}{x-4} \cdot \frac{\sqrt{x}+2}{\sqrt{x}+2} = \frac{x-4}{(x-4)(\sqrt{x}+2)} = \frac{1}{\sqrt{x}+2}$.

$\lim\limits_{x\to 4} \frac{1}{\sqrt{4}+2} = \frac{1}{2+2} = \frac{1}{4}$.

$$\boxed{\frac{1}{4}}$$

**(b)** Multiply by conjugate:

$\left(\sqrt{x^2+3x} - x\right) \cdot \frac{\sqrt{x^2+3x}+x}{\sqrt{x^2+3x}+x} = \frac{x^2+3x-x^2}{\sqrt{x^2+3x}+x} = \frac{3x}{\sqrt{x^2+3x}+x}$.

Divide numerator and denominator by $x$:

$\frac{3}{\sqrt{1+\frac{3}{x}}+1} \to \frac{3}{\sqrt{1+0}+1} = \frac{3}{2}$.

$$\boxed{\frac{3}{2}}$$

**(c)** Factor:

$\frac{(x-3)(x+3)}{(x-3)(x-2)} = \frac{x+3}{x-2}$ for $x \neq 3$.

$\lim\limits_{x\to 3} \frac{x+3}{x-2} = \frac{6}{1} = 6$.

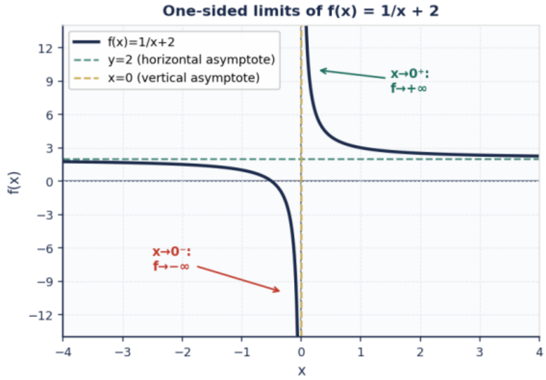
$$\boxed{6}$$

**(d)** Divide by $x^2$:

$\frac{3-\frac{2}{x}}{1+\frac{1}{x^2}} \to \frac{3-0}{1+0} = 3$.

$$\boxed{3}$$

## Problem 16 :



### Answer:

**(a)** As $x \to 0^-$, $f(x) \to -\infty$.
As $x \to 0^+$, $f(x) \to +\infty$.

$$\boxed{\lim_{x\to 0^-} f(x) = -\infty,\ \lim_{x\to 0^+} f(x) = +\infty}$$

**(b)** The two-sided limit does not exist because the left-hand and right-hand limits are not equal (one is $-\infty$, the other $+\infty$).

$$\boxed{\text{No two-sided limit}}$$

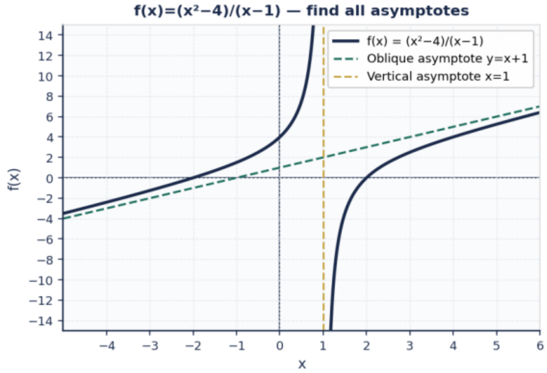
**(c)** $f(x) = 2 + \frac{1}{x}$. As $x \to \pm\infty$, $\frac{1}{x} \to 0$, so $f(x) \to 2$.
Horizontal asymptote: $y = 2$.

$$\boxed{y = 2}$$

**(d)** $2 + \frac{1}{x} > 3 \Rightarrow \frac{1}{x} > 1 \Rightarrow \frac{1-x}{x} > 0$.
Critical values: $x = 0$, $x = 1$.
Sign table shows positive on $(0، 1)$.

$$\boxed{(0، 1)}$$

## Problem 17 :



**Answer:**

**(a)** Domain: denominator not zero ⇒ $x \neq 1$.

$$\boxed{\mathbb{R} \setminus \{1\}}$$

**(b)** Divide $x^2 - 4$ by $x - 1$:
$(x-1)(x+1) = x^2 - 1$.
$x^2 - 4 = (x-1)(x+1) - 3$.
So $f(x) = x + 1 - \frac{3}{x-1}$.

**(c)** Vertical asymptote: $x = 1$ (denominator zero).
Oblique asymptote: $y = x + 1$ (since degree numerator = degree denominator + 1).

$$\boxed{x = 1, y = x + 1}$$

**(d)** $f(x) - (x+1) = -\frac{3}{x-1}$.
For $x > 1$, $-\frac{3}{x-1} < 0$, so the curve is **below** the oblique asymptote.
For $x < 1$, $-\frac{3}{x-1} > 0$, so the curve is **above** the oblique asymptote.

$$\boxed{\text{Below for } x > 1\text{, above for } x < 1}$$

## Problem 18 :

**Answer:**

**(a)** Factor: $\frac{(x-1)(x+1)}{(x-2)(x+2)} \geq 0$.

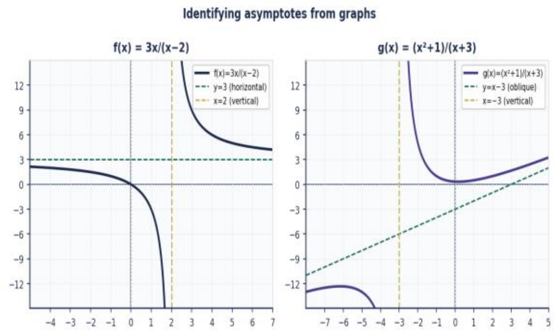
**(b)** Critical values: $x = -2, -1, 1, 2$ (ordered: $-2, -1, 1, 2$).
Exclude $x = \pm 2$ (denominator zero).

**(c)** Sign table:

| Interval | $x+2$ | $x+1$ | $x-1$ | $x-2$ | Product |
|---|---|---|---|---|---|
| $]-\infty,-2]$ | $-$ | $-$ | $-$ | $-$ | $+$ |
| $[-2,-1]$ | $+$ | $-$ | $-$ | $-$ | $-$ |
| $[-1,1]$ | $+$ | $+$ | $-$ | $-$ | $+$ |
| $[1,2]$ | $+$ | $+$ | $+$ | $-$ | $-$ |
| $[2,\infty[$ | $+$ | $+$ | $+$ | $+$ | $+$ |

**(d)** We want $\geq 0$: $]-\infty,-2] \cup [-1,1] \cup [2,\infty[$.

## Problem 19 :



**Answer:**

**(a)** $f(x) = \frac{3x}{x-2}$.
Vertical asymptote: $x = 2$. Horizontal asymptote: $y = 3$.
Algebraically: $f(x) = 3 + \frac{6}{x-2}$ (since $\frac{3x}{x-2} = 3 + \frac{6}{x-2}$).
As $x \to \pm\infty$, $f(x) \to 3$. ✓

**(b)** $g(x) = \frac{x^2+1}{x+3}$.
Degree numerator (2) = degree denominator (1) + 1 → oblique asymptote.
Divide: $x^2 + 1 = (x+3)(x-3) + 10$.
So $g(x) = x - 3 + \frac{10}{x+3}$.
Oblique asymptote: $y = x - 3$.

$$\boxed{\text{Oblique: } y = x - 3}$$

**(c)** $f(x) = 3 + \frac{6}{x-2}$. For $x > 2$, $\frac{6}{x-2} > 0$, so $f(x) > 3$ (above asymptote).
$g(x) = x - 3 + \frac{10}{x+3}$. For $x > -3$, $\frac{10}{x+3} > 0$, so $g(x) > x - 3$ (above asymptote).

$$\boxed{f \text{ above for } x > 2,\ g \text{ above for } x > -3}$$

**(d)** $f(x) = 3$: $3 + \frac{6}{x-2} = 3 \Rightarrow \frac{6}{x-2} = 0 \Rightarrow$ no solution.
$g(x) = x - 3$: $x - 3 + \frac{10}{x+3} = x - 3 \Rightarrow \frac{10}{x+3} = 0 \Rightarrow$ no solution.

$$\boxed{\text{No intersections with asymptotes}}$$

## Problem 20 :

**Answer:**

**(a)** Numerator: $x^2 - 5x + 6 = (x-2)(x-3)$.
Denominator: $x^2 - 9 = (x-3)(x+3)$.

$$\boxed{f(x) = \frac{(x-2)(x-3)}{(x-3)(x+3)}}$$

**(b)** Undefined where denominator is zero: $x = 3$ and $x = -3$.
At $x = 3$, numerator is also zero → **removable discontinuity**.
At $x = -3$, numerator is not zero $(-5 \neq 0)$ → vertical asymptote.

**(c)** $f(x) = \frac{x-2}{x+3}$ for $x \neq 3$ (and $x \neq -3$).

$$\boxed{f(x) = \frac{x-2}{x+3}, x \neq 3, \ -3}$$

**(d)** $\lim_{x\to 3} f(x) = \lim_{x\to 3}\frac{x-2}{x+3} = \frac{1}{6}$.
$\lim_{x\to -3} f(x)$: denominator approaches 0, numerator approaches $-5$, so the limit does not exist (goes to $\pm\infty$).

$$\boxed{\lim_{x\to 3} = \frac{1}{6},\ \lim_{x\to -3} \text{ does not exist}}$$

## Problem 21 :

### Answer:

**Proof:** For $x > 0$, $0 < \frac{1}{x^n} \le \frac{1}{x}$.
As $x \to +\infty$, $\frac{1}{x} \to 0$. By the squeeze theorem, $\frac{1}{x^n} \to 0$. ✓

**(a)** $\left(1+\frac{1}{x}\right)^2 \to (1+0)^2 = 1$.

$$\boxed{1}$$

**(b)** $\frac{3x^2-2}{x^2+1} - 3 = \frac{3x^2-2-3x^2-3}{x^2+1} = \frac{-5}{x^2+1} \to 0$.

$$\boxed{0}$$

**(c)** $x\left(\frac{1}{x} - \frac{1}{x^2}\right) = 1 - \frac{1}{x} \to 1$.

$$\boxed{1}$$

## Problem 22 :

### Answer:

**(a)** Denominator: $x^2 - x - 6 = (x-3)(x+2)$.
Domain: $x \ne 3,\ -2$.

$$\boxed{\mathbb{R} \setminus \{3, -2\}}$$

**(b)** Vertical asymptotes: $x = 3$, $x = -2$ (numerator not zero at these points).
Horizontal asymptote: degree numerator (1) < degree denominator (2) ⇒ $y = 0$.

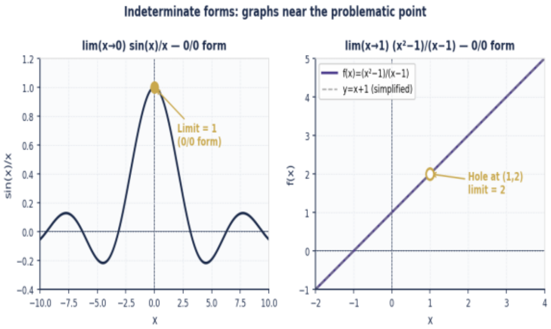
$$\boxed{x = 3,\ x = -2,\ y = 0}$$

**(c)** At $x = 3$: numerator $= 2(3) + 3 = 9 \ne 0$.
As $x \to 3^+$: denominator $\to 0^+$, so $f(x) \to +\infty$.
As $x \to 3^-$: denominator $\to 0^-$, so $f(x) \to -\infty$.
At $x = -2$: numerator $= 2(-2) + 3 = -1 \ne 0$.
As $x \to -2$: denominator $\to 0$, so $f(x) \to \pm\infty$ (no finite limit).

$$\boxed{\lim_{x\to 3^+} = +\infty,\ \lim_{x\to 3^-} = -\infty,\ \lim_{x\to -2} \text{ does not exist}}$$

**(d)** Critical values: numerator zero at $x = -\frac{3}{2}$, denominator zeros at $x = -2{,}3$.

## Problem 23 :



Indeterminate forms: graphs near the problematic point

lim(x→0) sin(x)/x — 0/0 form

lim(x→1) (x²−1)/(x−1) — 0/0 form

### Answer:

**(a)** From the graph, $\lim_{x\to 0}\frac{\sin x}{x} = 1$ (standard result).

$$\boxed{1}$$

**(b)** $\frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1} = x + 1$ for $x \ne 1$.
$\lim_{x\to 1}(x+1) = 2$. Hole at $(1, 2)$.

$$\boxed{2}$$

**(c)** Let $u = 3x$. As $x \to 0$, $u \to 0$.
$\frac{\sin(3x)}{3x} = \frac{\sin u}{u} \to 1$.

$$\boxed{1}$$

**(d)** Factor: $x^n - a^n = (x-a)(x^{n-1} + ax^{n-2} + \cdots + a^{n-1})$.
$\frac{x^n - a^n}{x-a} \to na^{n-1}$.

$$\boxed{na^{n-1}}$$

## Problem 24 :

### Answer:

**(a)** Left: $\lim_{x\to 1^-}(x^2 - 2) = 1 - 2 = -1$.
Right: $\lim_{x\to 1^+}(3x - 4) = 3 - 4 = -1$.

$$\boxed{\lim_{x\to 1^-} = -1,\ \lim_{x\to 1^+} = -1}$$

**(b)** $f(1) = 3(1) - 4 = -1$.
Both one-sided limits equal $f(1)$,
so $f$ is **continuous** at $x = 1$.

$$\boxed{\text{Continuous}}$$

**(c)** For $x < 1$: $x^2 - 2 > 0 \Rightarrow x^2 > 2 \Rightarrow x < -\sqrt{2}$ or $x > \sqrt{2}$.
Combined with $x < 1$: $x < -\sqrt{2}$ or $\sqrt{2} < x < 1$.
For $x \ge 1$: $3x - 4 > 0 \Rightarrow x > \frac{4}{3}$.
Combined: $x \ge \frac{4}{3}$.
Full solution: $\left(-\infty, -\sqrt{2}\right) \cup \left(\sqrt{2}, 1\right) \cup \left(\frac{4}{3}, \infty\right)$.

$$\boxed{\left(-\infty, -\sqrt{2}\right) \cup \left(\sqrt{2}, 1\right) \cup \left(\frac{4}{3}, \infty\right)}$$

**(d)** Need left limit equal to right limit at $x = 1$:
$\lim_{x\to 1^-}(x^2 + c) = 1 + c$.
Right limit: $3(1) - 4 = -1$.
Set $1 + c = -1 \Rightarrow c = -2$.

$$\boxed{c = -2}$$

## Problem 25 :

### Answer:

**(a)** Factor: $\frac{(x-2)(x+2)}{(x+3)(x-2)}=\frac{x+2}{x+3}$ for $x\neq 2$.

**(b)** Critical values: $x=-3$ (denominator zero, exclude), $x=-2$ (numerator zero), $x=2$ (excluded, cancelled factor).

**(c)** Sign table for $\frac{x+2}{x+3}$:

| Interval | $x+2$ | $x+3$ | Quotient |
|---|---|---|---|
| $]-\infty,-3]$ | $-$ | $-$ | $+$ |
| $[-3,-2]$ | $-$ | $+$ | $-$ |
| $[-2,+\infty[$ | $+$ | $+$ | $+$ |

**(d)** We want $>0$: $]-\infty,-3]\cup[-2,+\infty[$.
Exclude $x=2$ from the solution (since original is undefined there).
Final: $]-\infty,-3]\cup[-2,2]\cup[2,+\infty[$.

$$\boxed{]-\infty,-3]\cup[-2,2]\cup[2,+\infty[}$$

## Problem 26 :

### Answer:

**(a)** Divide numerator and denominator by $x^2$ (highest power): $\frac{3+\frac{5}{x}-\frac{2}{x^2}}{2-\frac{4}{x}+\frac{1}{x^2}}$.

**(b)** Apply the limit term by term: $\lim_{x\to+\infty}\frac{1}{x}=0$ and $\lim_{x\to+\infty}\frac{1}{x^2}=0$.
So the expression becomes $\frac{3+0-0}{2-0+0}=\frac{3}{2}$.

**(c)** As $x\to-\infty$, $1/x\to 0$ and $1/x^2\to 0$ as well.
So the limit is the same: $\frac{3}{2}$.

$$\boxed{\frac{3}{2}}$$

## Problem 27 :

### Answer:

**(a)** Bring to one side: $x+\frac{1}{x-1}-3\geq 0$.

**(b)** Common denominator: $\frac{x(x-1)+1-3(x-1)}{x-1}\geq 0$
$=\frac{x^2-x+1-3x+3}{x-1}=\frac{x^2-4x+4}{x-1}$.

**(c)** Factor numerator: $\frac{(x-2)^2}{x-1}\geq 0$.

**(d)** The numerator $(x-2)^2\geq 0$ always.
The sign depends only on the denominator $x-1$:

- $x>1$: quotient $\geq 0$ (zero at $x=2$).
- $x<1$: quotient $<0$ (since numerator positive, denominator negative).
  Solution: $x>1$ (including $x=2$).

$$\boxed{(1,\infty)}$$

## Problem 28 :

### Answer:

**(a)** As $x\to 2^+$, $x-2\to 0^+$, so $\frac{1}{x-2}\to+\infty$.

$$\boxed{+\infty}$$

**(b)** As $x\to 2^-$, $x-2\to 0^-$, so $\frac{1}{x-2}\to-\infty$.

$$\boxed{-\infty}$$

**(c)** Divide by $x^2$: $\frac{x-\frac{1}{x}}{1+\frac{1}{x^2}}\to\frac{x-0}{1+0}=x\to+\infty$.

$$\boxed{+\infty}$$

**(d)** $\frac{1}{x}-\frac{1}{x^2}=\frac{x-1}{x^2}$.
As $x\to 0^+$, numerator $\to -1$, denominator $\to 0^+$, so limit $\to-\infty$.

$$\boxed{-\infty}$$

## Problem 29 :

### Answer:

**(a)** Vertical asymptote: denominator zero $\Rightarrow cx+d=0\Rightarrow x=-\frac{d}{c}$.
Numerator at this point: $a\left(-\frac{d}{c}\right)+b=\frac{-ad+bc}{c}=\frac{bc-ad}{c}\neq 0$ since $ad-bc\neq 0$.
So $x=-\frac{d}{c}$ is a vertical asymptote.

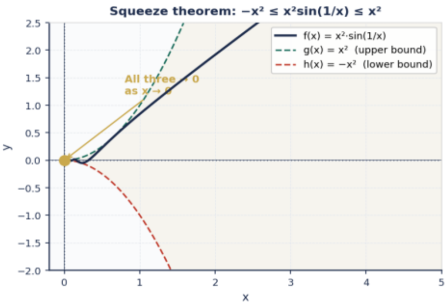
**(b)** Horizontal asymptote: as $x\to\pm\infty$, $f(x)=\frac{ax+b}{cx+d}\to\frac{a}{c}$ (divide numerator and denominator by $x$).
So $y=\frac{a}{c}$ is the horizontal asymptote.

**(c)** Rewrite: $f(x)=\frac{a}{c}+\frac{bc-ad}{c(cx+d)}$.
This shows that $f$ is a scaled version of $\frac{1}{cx+d}$ plus a constant.
As $x\to\pm\infty$, the second term vanishes, giving the horizontal asymptote.
Near $x=-\frac{d}{c}$, the second term blows up, giving the vertical asymptote.

**(d)** If $ad-bc=0$, then $bc=ad$, so $b=\frac{ad}{c}$.
Then $f(x)=\frac{ax+\frac{ad}{c}}{cx+d}=\frac{a(cx+d)}{c(cx+d)}=\frac{a}{c}$ (constant).
A constant function has no asymptotes.

$$\boxed{\text{If } ad-bc=0,\ f \text{ is constant}}$$

## Problem 30 :



### Answer:

**(a)** $\sin\left(\frac{1}{x}\right)$ is always between $-1$ and $1$: $-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$ for all $x \neq 0$.

**(b)** Multiply by $x^2 \geq 0$: $-x^2 \leq x^2\sin\left(\frac{1}{x}\right) \leq x^2$.

**(c)** As $x \to 0$, $-x^2 \to 0$ and $x^2 \to 0$.
By the squeeze theorem, $x^2\sin\left(\frac{1}{x}\right) \to 0$.

$$\boxed{0}$$

**(d)** For $x\sin\left(\frac{1}{x}\right)$, we have $-|x| \leq x\sin\left(\frac{1}{x}\right) \leq |x|$.
Since $|x| \to 0$, the squeeze theorem gives $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$.

For $\sin\left(\frac{1}{x}\right)$ alone: it oscillates between $-1$ and $1$, so no limit exists.

$$\boxed{x\sin(1/x) \to 0, \sin(1/x) \text{ has no limit}}$$

## Problem 31 :

**Answer:**

**(a)** Let $u = x^2$. Then $u^2 - 10u + 9 = (u-1)(u-9) \leq 0$.
So $1 \leq u \leq 9 \Rightarrow 1 \leq x^2 \leq 9$.

**(b)** $x^2 \geq 1 \Rightarrow |x| \geq 1 \Rightarrow x \leq -1$ or $x \geq 1$.
$x^2 \leq 9 \Rightarrow |x| \leq 3 \Rightarrow -3 \leq x \leq 3$.
Intersection: $[-3,-1] \cup [1,3]$.

**(c)** Sign table confirms: positive outside $[-3,-1] \cup [1,3]$, zero at roots $\pm 1, \pm 3$, negative between.

$$\boxed{[-3,-1] \cup [1,3]}$$

## Problem 32 :

**Answer:**

**(a)** $f(x) = 3 + \frac{1}{x}$.
As $x \to +\infty$, $f(x) \to 3$.
But $3 + \frac{1}{x} = 3$ has no solution (since $\frac{1}{x} \neq 0$ for finite $x$).
So $f$ never equals $3$.

**(b)** The graph approaches the horizontal line $y = 3$ asymptotically but never touches it.

**(c)** A horizontal asymptote means the curve gets arbitrarily close to the line as $x \to \pm\infty$, but it does **not** have to touch or cross it.

**(d)** Example where the limit value is achieved: $f(x) = 3$ (constant function) reaches $3$ everywhere.
Or $f(x) = 3 + \frac{\sin x}{x}$ reaches $3$ at infinitely many points (e.g., $x = n\pi$ with $\sin(n\pi) = 0$).

$$\boxed{\text{Counterexample: } f(x) = 3 + \frac{1}{x}}$$

## Problem 33 :

**Answer:** **(a)** Divide $x^3 + x$ by $x^2 - 1$:
$(x^2-1)(x) = x^3 - x$.
$x^3 + x = (x^2-1)(x) + 2x$.
So $f(x) = x + \frac{2x}{x^2-1}$.

**(b)** Vertical asymptotes: denominator zero $\Rightarrow x^2 - 1 = 0 \Rightarrow x = \pm 1$.
Oblique asymptote: $y = x$ (since quotient is $x$).

**(c)** $f(x) - x = \frac{2x}{x^2-1}$.
For $x > 1$: numerator positive, denominator positive $\Rightarrow$ curve **above** the oblique asymptote.
For $0 < x < 1$: numerator positive, denominator negative $\Rightarrow$ curve **below**.
For $-1 < x < 0$: numerator negative, denominator negative $\Rightarrow$ curve **above**.
For $x < -1$: numerator negative, denominator positive $\Rightarrow$ curve **below**.

**(d)** $f(x) = x$ when $\frac{2x}{x^2-1} = 0 \Rightarrow x = 0$.
The curve crosses its oblique asymptote at the origin $(0,0)$.

$$\boxed{y = x \text{ (oblique)}, x = 1,\ x = -1 \text{ (vertical)}}$$

## Problem 34 :

**Answer:** **(a)** Bring to one side: $\frac{x^2-4}{x^2-2x-3} - 1 \geq 0$
$= \frac{x^2-4-(x^2-2x-3)}{x^2-2x-3} = \frac{2x-1}{x^2-2x-3} \geq 0$.

**(b)** Factor denominator: $x^2 - 2x - 3 = (x-3)(x+1)$.
Critical values: numerator zero at $x = \frac{1}{2}$, denominator zeros at $x = 3,\ -1$ (exclude).

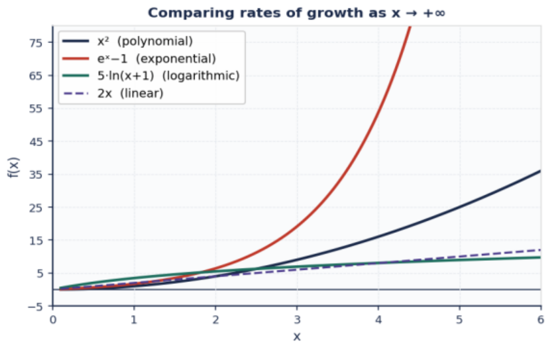
**(c)** Sign table:

| Interval | $2x-1$ | $x+1$ | $x-3$ | Product |
|---|---|---|---|---|
| $]-\infty,-1]$ | $-$ | $-$ | $-$ | $-$ |
| $[-1,\frac{1}{2}]$ | $-$ | $+$ | $-$ | $+$ |
| $[\frac{1}{2},3]$ | $+$ | $+$ | $-$ | $-$ |
| $[3,+\infty[$ | $+$ | $+$ | $+$ | $+$ |

**(d)** We want $\geq 0$: $[-1,\frac{1}{2}] \cup [3,+\infty[$.

$$\boxed{\left[-1,\frac{1}{2}\right] \cup [3,+\infty[}$$

## Problem 35 :



**Answer:**

**(a)** Slowest to fastest: $\ln x < \sqrt{x} < x^2 < e^x$.

$$\boxed{\ln x,\ \sqrt{x},\ x^2,\ e^x}$$

**(b)** From the graph, $e^x$ overtakes $x^2$ at approximately $x \approx 5$–$6$.

$$\boxed{x \approx 5.5}$$

**(c)** $\lim_{x\to+\infty} \frac{x^2}{e^x} = 0$ (exponential dominates any polynomial).

$$\boxed{0}$$

**(d)** $\lim_{x\to+\infty} \frac{\ln x}{x} = 0$ (any positive power of $x$ grows faster than $\ln x$).

$$\boxed{0}$$

# Problems 36–50 — Difficult & Real-World Applications

## Problem 36 :

**Answer:**

**(a)** $\left| \frac{x+1}{x} - 1 \right| = \left| \frac{1}{x} \right| = \frac{1}{x}$ for $x > 0$.

**(b)** Given $\varepsilon > 0$, choose $M = \frac{1}{\varepsilon}$.
Then for all $x > M$, we have $x > \frac{1}{\varepsilon} \Rightarrow \frac{1}{x} < \varepsilon$.
So $\left| \frac{x+1}{x} - 1 \right| < \varepsilon$.

**(c)** Complete proof:
For any $\varepsilon > 0$, let $M = \frac{1}{\varepsilon}$. Then for all $x > M$:
$\left| \frac{x+1}{x} - 1 \right| = \frac{1}{x} < \varepsilon$.
Thus $\lim_{x\to+\infty} \frac{x+1}{x} = 1$. ✓

**(d)** The formal proof provides a constructive bound: for any required precision $\varepsilon$, it gives a specific threshold $M$. The informal argument "the $/x$ terms vanish" only conveys intuition.

## Problem 37 :

**Answer:**

**(a)** $x^3 > x^2 \Rightarrow x^2(x-1) > 0$.
Since $x^2 > 0$ for $x > 0$, this is true when $x > 1$.
Threshold: $x = 1$.

**(b)** For $0 < x < 1$: $x^3 < x^2$ (since multiplying by $x < 1$ decreases the value).
So $f(x) = x^2$ is larger.

**(c)** At $x = 1$, $f(1) = g(1) = 1$.
For $x > 1$, $x^3 > x^2$.

**(d)** In general, for $n > m \geq 1$, $x^n > x^m$ when $x > 1$.

$$\boxed{x > 1}$$

.

## Problem 38 :

**Answer:**

**(a)** Let $h(x) = x - 1 - \ln x$.
$h'(x) = 1 - \frac{1}{x}$.
$h'(x) = 0$ at $x = 1$.
$h(1) = 0$. Since $h' < 0$ for $0 < x < 1$ and $h' > 0$ for $x > 1$, $h$ has a minimum at $x = 1$.
Thus $h(x) \geq 0$ for all $x > 0 \Rightarrow x - 1 \geq \ln x$. ✓

**(b)** Let $t = \sqrt{x} > 0$. Then $x = t^2$.
$\sqrt{x} \leq \frac{x+1}{2} \Rightarrow t \leq \frac{t^2+1}{2} \Rightarrow 2t \leq t^2 + 1 \Rightarrow t^2 - 2t + 1 \geq 0 \Rightarrow (t-1)^2 \geq 0$, which is always true. ✓

**(c)** The function $f(t) = \frac{1}{t}$ is decreasing on $(0,\infty)$.
For $t \in [x, x+1]$, the area under the curve lies between the areas of two rectangles:
$\frac{1}{x+1} \cdot 1 \leq \int_x^{x+1} \frac{1}{t}\,dt \leq \frac{1}{x} \cdot 1$.
$\int_x^{x+1} \frac{1}{t}\,dt = \ln(x+1) - \ln x$.
Thus $\frac{1}{x+1} \leq \ln(x+1) - \ln x \leq \frac{1}{x}$. ✓

**(d)** Sum from $x = 1$ to $n$:
$\sum_{k=1}^{n} \frac{1}{k+1} < \sum_{k=1}^{n} (\ln(k+1) - \ln k) < \sum_{k=1}^{n} \frac{1}{k}$.
The middle sum telescopes: $\ln(n+1) - \ln 1 = \ln(n+1)$.
Thus $\ln(n+1) < 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. ✓

$$\boxed{\ln(n+1) < 1 + \frac{1}{2} + \cdots + \frac{1}{n}}$$

.

## Problem 39 :

**Answer:** Simplify the expression (valid for $x \neq \pm k$):
$\frac{(x-k)(x+2k)}{(x-k)(x+k)} = \frac{x+2k}{x+k}$, provided $x \neq k$.
We require $\frac{x+2k}{x+k} > 0$ for every $x > 2k$.

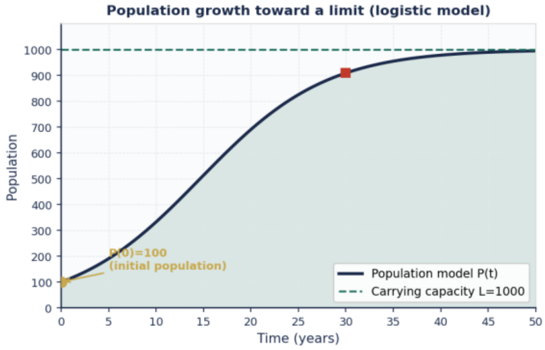
Case 1: $k > 0$
For $x > 2k$, we have $x + 2k > 0$ and $x + k > 0$.
Their quotient is positive. ✓

Case 2: $k = 0$
The expression becomes $\frac{x}{x} = 1$ for $x \neq 0$. The condition is $x > 0$, so $1 > 0$ holds. ✓

Case 3: $k < 0$
Choose $x = -k$ (which is $> 2k$ because $-k > 2k \Rightarrow -k - 2k > 0 \Rightarrow -3k > 0 \Rightarrow k < 0$, true).
At $x = -k$, the denominator $x + k = 0$, so the expression is undefined. Thus the inequality cannot hold for all $x > 2k$.

Therefore the required values are $k \geq 0$.
Answer: $\boxed{k \geq 0}$

## Problem 40 :



### Answer:

**(a)** $P(0) = \frac{1000}{1+9} = 100$ birds.
Initial population: 100 birds.

**(b)** As $t \to +\infty$, $e^{-0.15t} \to 0$, so $P(t) \to \frac{1000}{1+0} = 1000$.
Carrying capacity = 1000 birds.

**(c)** $P(t) = 500$: $\frac{1000}{1+9e^{-0.15t}} = 500 \Rightarrow 1 + 9e^{-0.15t} = 2 \Rightarrow 9e^{-0.15t} = 1 \Rightarrow e^{-0.15t} = \frac{1}{9}$.

$-0.15t = \ln\left(\frac{1}{9}\right) = -\ln 9$.

$t = \frac{\ln 9}{0.15} \approx 14.65$ years.

$$\boxed{t = \frac{\ln 9}{0.15} \approx 14.65 \text{ years}}$$

**(d)** 90% of 1000 = 900.
$P(t) = 900 \Rightarrow \frac{1000}{1+9e^{-0.15t}} = 900 \Rightarrow 1 + 9e^{-0.15t} = \frac{1000}{900} = \frac{10}{9}$.
$9e^{-0.15t} = \frac{1}{9} \Rightarrow e^{-0.15t} = \frac{1}{81}$.
$t = \frac{\ln 81}{0.15} = \frac{4\ln 3}{0.15} \approx 29.3$ years.

$$\boxed{t \approx 29.3 \text{ years}}$$

## Problem 41 :

### Answer:

**(a)** Evaluate $f(x) = x^3 - 3x + 1$ at convenient points:

$$f(-2) = -8 + 6 + 1 = -1 < 0$$
$$f(-1) = -1 + 3 + 1 = 3 > 0$$
$$f(0) = 1 > 0$$
$$f(1) = 1 - 3 + 1 = -1 < 0$$
$$f(2) = 8 - 6 + 1 = 3 > 0$$

**(b)** Sign changes:

- On $[-2, -1]$: from $-$ to $+$ $\rightarrow$ one root.
- On $[-1, 1]$: from $+$ to $-$ $\rightarrow$ one root (actually there is a sign change between 0 and 1: $f(0) = 1$, $f(1) = -1$).
  Wait: check $[-1, 0]$: $f(-1) = 3$, $f(0) = 1$ (no sign change).
  Check $[0, 1]$: $f(0) = 1$, $f(1) = -1 \rightarrow$ sign change.
- On $[1, 2]$: from $-$ to $+$ $\rightarrow$ one root.

So three roots in $[-2, -1]$, $[0, 1]$, $[1, 2]$.

**(c)** $f'(x) = 3x^2 - 3 = 3(x-1)(x+1)$.
Critical points at $x = -1$ (local max, $f(-1) = 3$) and $x = 1$ (local min, $f(1) = -1$).
Since the local max is positive and the local min is negative, there are exactly three roots.

**(d)** The function is monotone on each interval, so exactly one root in each.

$$\boxed{\text{Three real roots in } [-2, -1],\ [0,1],\ [1,2]}$$

## Problem 42 :

### Answer:

**(a)** Values:
$x = 1$: $1^1 = 1$
$x = 0.5$: $0.5^{0.5} = e^{0.5\ln 0.5} \approx e^{-0.3466} \approx 0.707$
$x = 0.1$: $0.1^{0.1} = e^{0.1\ln 0.1} \approx e^{-0.2303} \approx 0.794$
$x = 0.01$: $0.01^{0.01} \approx e^{-0.04605} \approx 0.955$
$x = 0.001$: $0.001^{0.001} \approx e^{-0.00691} \approx 0.993$
Limit appears to be 1.

**(b)** $x^x = e^{x\ln x}$.
As $x \to 0^+$, $x\ln x \to 0$ (standard result: $\ln x$ grows slower than $1/x$ decays).
So $x\ln x \to 0$.

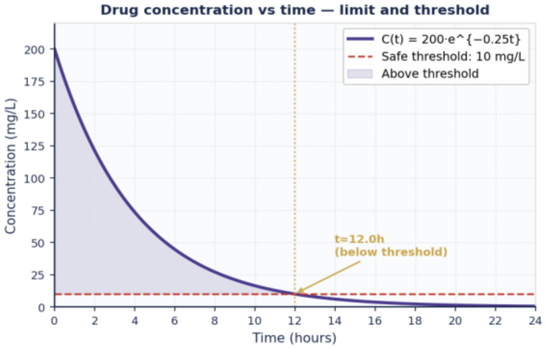
**(c)** $\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x\ln x} = e^0 = 1$.

$$\boxed{1}$$

**(d)** Minimum of $y = x^x$:
$\ln y = x\ln x$. Differentiate: $\frac{d}{dx}(x\ln x) = \ln x + 1$.
Set to 0: $\ln x = -1 \Rightarrow x = e^{-1} = \frac{1}{e}$.
Minimum value: $\left(\frac{1}{e}\right)^{1/e} = e^{-1/e} \approx 0.6922$.

$$\boxed{\text{Min } e^{-\frac{1}{e}} \text{ at } x = e^{-1}}$$

## Problem 43



### Answer:

**(a)** As $t \to +\infty$, $C(t) \to 0$.
The drug is completely eliminated.

**(b)** $200e^{-0.25t} = 10 \Rightarrow e^{-0.25t} = \frac{1}{20} \Rightarrow t = \frac{\ln 20}{0.25} = 4\ln 20 \approx 11.98$ hours.

$$\boxed{t = 4\ln 20 \approx 12 \text{ hours}}$$

**(c)** $200e^{-0.25t} \geq 50 \Rightarrow e^{-0.25t} \geq \frac{1}{4} \Rightarrow -0.25t \geq \ln\left(\frac{1}{4}\right) \Rightarrow -0.25t \geq -\ln 4 \Rightarrow t \leq 4\ln 4 \approx$ 5.55 hours.

$$\boxed{\text{Effective for } 0 \leq t \leq 4\ln 4 \text{ hours}}$$

**(d)** $200e^{-0.25t} = 30 \Rightarrow e^{-0.25t} = \frac{3}{20} \Rightarrow t = 4\ln\left(\frac{20}{3}\right) \approx$ 7.64 hours.
In 12 hours, doses at $t = 0$ and $t \approx 7.64$: 2 doses.
Cost: $2 \times 500 = 1000$ DA.

$$\boxed{t = 4\ln(20/3) \approx 7.64 \text{ h, cost } = 1000 \text{ DA}}$$

## Problem 44 :

### Answer:

**(a)** $| f(x) + g(x) - (L + M) | \leq | f(x) - L | + | g(x) - M |$.
Given $\varepsilon > 0$, choose $M_1$ so $| f - L | < \varepsilon/2$ for $x > M_1$, and $M_2$ so $| g - M | < \varepsilon/2$ for $x > M_2$.
For $x > \max(M_1, M_2)$, the sum is $< \varepsilon$. ✓

**(b)** $| cf(x) - cL | = | c || f(x) - L |$.
If $c \neq 0$, choose $M$ so $| f - L | < \varepsilon/| c |$.
If $c = 0$, trivial. ✓

**(c)**

$$\begin{aligned} | f(x)g(x) - LM | = & \\ & | f(x)(g(x) - M) + (f(x) - L)M \\ & | \leq | f(x) || g(x) - M | + | M | \\ & | f(x) - L |. \end{aligned}$$

Since $f(x)$ is bounded near $+\infty$ (approaches $L$), take $| f(x) | \leq | L | + 1$ eventually.
Choose $M_1, M_2$ to make each term small. ✓

**(d)** By induction: base $n = 1$ trivial.
Assume $\lim f^{n-1} = L^{n-1}$.
Then $\lim f^n = \lim(f^{n-1} \cdot f) = (\lim f^{n-1})(\lim f) = L^{n-1} \cdot L = L^n$. ✓

## Problem 45 :

### Answer:

**(a)** Choose $\varepsilon = \frac{L}{2} > 0$.
By the definition of the limit, there exists $\delta > 0$ such that for all $x$ with $0 < | x - a | < \delta$, $| f(x) - L | < \frac{L}{2}$.

**(b)** Then $L - \frac{L}{2} < f(x) < L + \frac{L}{2} \Rightarrow \frac{L}{2} < f(x) < \frac{3L}{2}$.
Thus $f(x) > \frac{L}{2} > 0$ on the punctured interval $(a - \delta, a + \delta) \setminus \{a\}$.

**(c)** If $\lim f = L$ and $\lim g = M$ with $L > M$, then $\lim(f - g) = L - M > 0$.
By the previous result, there is an interval around $a$ where $f(x) - g(x) > 0$, i.e., $f(x) > g(x)$.

**(d)** Counterexample for $L = M = 0$:
$f(x) = \frac{\sin x}{x}$ and $g(x) = -\frac{\sin x}{x}$ both approach 0, but neither is consistently greater than the other.

$$\boxed{\text{Works for } L > 0\text{, fails for } L = 0}$$

## Problem 46 :

### Answer:

**(a)** $d(0) = 0(64 - 0 + 0) = 0$.
$d(4) = 4(64 - 8(16) + 64) = 4(64 - 128 + 64) = 4(0) = 0$.
Confirmed.

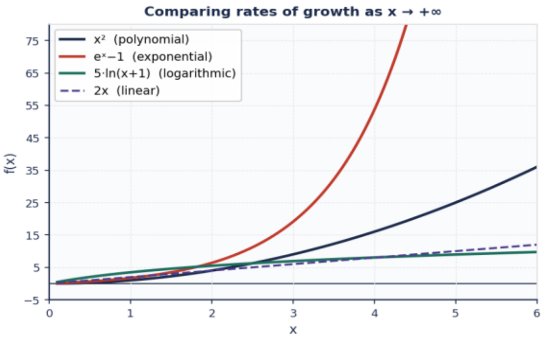
**(b)** Expand: $d(x) = 64x - 8x^3 + x^4$.
$d'(x) = 64 - 24x^2 + 4x^3$.
Set $d'(x) = 0$: $4x^3 - 24x^2 + 64 = 0 \Rightarrow$ divide by 4: $x^3 - 6x^2 + 16 = 0$.
Test $x = 2$: $8 - 24 + 16 = 0$. So $x = 2$ is a critical point.
Factor: $(x - 2)(x^2 - 4x - 8) = 0$.
The quadratic gives $x = 2 \pm 2\sqrt{3}$. Only $x = 2 + 2\sqrt{3} \approx 5.46$ is outside $[0, 4]$.
So critical point in the interval: $x = 2$.

**(c)** $d(2) = 2(64 - 8(4) + 8) = 2(64 - 32 + 8) = 2(40) = 80$ units.
Maximum deflection = 80 units.

**(d)** Since 80 mm > 5 mm, the beam does **not** meet the specification.

$$\boxed{\text{Does not meet specification}}$$

## Problem 47 :



### Answer:

**(a)** From the graph, $e^x$ overtakes $x^2$ near $x \approx 5$–6.

**(b)** $\lim\limits_{x\to+\infty} \frac{e^x}{x^2} = +\infty$ (exponential dominates).
$\lim\limits_{x\to+\infty} \frac{\ln x}{\sqrt{x}} = 0$ (log grows slower than any positive power).
$\lim\limits_{x\to+\infty} \frac{x^3}{e^x} = 0$ (exponential dominates).

**(c)** $e^x$ grows fastest. Then $x^{100}$ grows faster than $x^{10}$, but both are polynomial.
Order: $e^x \gg x^{100} \gg x^{10}$.

**(d)** $e^{\sqrt{x}}$ grows faster than any polynomial (since $e^{\sqrt{x}}/x^n \to +\infty$) but slower than $e^x$.

$$\boxed{e^{\sqrt{x}}}$$

## Problem 48 :

**Answer:**

**(a)** Numerator: $x^3 - x = x(x-1)(x+1)$.
Denominator: $x^2 - 4x + 4 = (x-2)^2$.

**(b)** Domain: $x \neq 2$.
At $x = 2$, numerator = $2(1)(3) = 6 \neq 0$, so $x = 2$ is a vertical asymptote.
No removable discontinuity.

**(c)** Divide: $x^3 - x = (x^2 - 4x + 4)(x + 4) + (15x - 16)$.
So $f(x) = x + 4 + \frac{15x-16}{(x-2)^2}$.
Oblique asymptote: $y = x + 4$.

**(d)** $f(x) - (x + 4) = \frac{15x-16}{(x-2)^2}$.
Denominator always positive (square).
Sign of numerator: $15x - 16 > 0 \Leftrightarrow x > \frac{16}{15}$.
So for $x > \frac{16}{15}$ (and $x \neq 2$), curve is **above** the asymptote.
For $x < \frac{16}{15}$ (and $x \neq 2$), curve is **below** the asymptote.

$$\boxed{\text{Vertical: } x = 2\text{, Oblique: } y = x + 4}$$

## Problem 49 :

**Answer:**

**(a)** $AC(q) = q^2 - 12q + 60 + \frac{100}{q}$.

**(b)** As $q \to +\infty$, $AC(q) \to q^2 - 12q + 60 \to +\infty$.
Long-run average cost is unbounded (due to cubic total cost).

**(c)** Already done in (a): $AC(q) = q^2 - 12q + 60 + \frac{100}{q}$.

**(d)** $AC'(q) = 2q - 12 - \frac{100}{q^2}$.
Set $AC'(q) = 0$: $2q - 12 - \frac{100}{q^2} = 0 \Rightarrow$ multiply by $q^2$: $2q^3 - 12q^2 - 100 = 0 \Rightarrow q^3 - 6q^2 - 50 = 0$.
Test $q = 7$: $343 - 294 - 50 = -1 \approx 0$. So minimum near $q = 7$.

$$\boxed{q \approx 7}$$

## Problem 50 :

**Answer: (a)** $\lim_{R\to+\infty} F(R) = \lim_{R\to+\infty} \frac{800R}{R+200} = 800$.
Maximum fox population (carrying capacity) is 800.

**(b)** $F(200) = \frac{800(200)}{400} = 400$.
$F(600) = \frac{800(600)}{800} = 600$.
Doubling rabbits from 200 to 400 does not double foxes → diminishing returns.

**(c)** $\frac{800R}{R+200} = \frac{800(R+200)-160000}{R+200} = 800 - \frac{160000}{R+200}$.
This is a hyperbolic function (Möbius transformation).

**(d)** $F(R) \geq 400$: $800 - \frac{160000}{R+200} \geq 400 \Rightarrow \frac{160000}{R+200} \leq 400 \Rightarrow R + 200 \geq 400 \Rightarrow R \geq 200$.

$$\boxed{R \geq 200}$$

**(e)** $F(0) = 0$. No rabbits means no foxes (the predator depends entirely on the prey).

$$\boxed{F(0) = 0}$$

### CHAPTER 4 — DERIVATIVES & DIFFERENTIATION

# SOLUTIONS OF PROBLEMS

## Solutions to Problems 1–10 — Instant Recall

### Problem 1 :

**Answer:**

**(a)** $f(x) = x^6$

$$f'(x) = 6x^{6-1} = 6x^5$$

$$\boxed{6x^5}$$

**(b)** $g(x) = 5x^3 - 2x^2 + 7x - 1$

$$g'(x) = 5(3x^2) - 2(2x) + 7(1) - 0$$
$$= 15x^2 - 4x + 7$$

$$\boxed{15x^2 - 4x + 7}$$

**(c)** $h(x) = \frac{4}{x^2} = 4x^{-2}$

$$h'(x) = 4(-2)x^{-3} = -\frac{8}{x^3}$$

$$\boxed{-\frac{8}{x^3}}$$

**(d)** $k(x) = \sqrt{x^5} = x^{5/2}$

$$k'(x) = \frac{5}{2}x^{5/2-1} = \frac{5}{2}x^{3/2}$$

$$\boxed{\frac{5}{2}x^{3/2}}$$

**(e)** $p(x) = x^{-3} + x^{1/3}$

$$p'(x) = -3x^{-4} + \frac{1}{3}x^{-2/3}$$

$$\boxed{-3x^{-4} + \frac{1}{3}x^{-2/3}}$$

### Problem 2 :

**Answer:**

**(a)** $f(x) = x^3 \sin x$

$$f'(x) = (x^3)'\sin x + x^3(\sin x)'$$
$$= 3x^2 \sin x + x^3 \cos x$$

$$\boxed{3x^2 \sin x + x^3 \cos x}$$

**(b)** $g(x) = (x^2 + 1)(3x - 2)$

$$g'(x) = (x^2 + 1)'(3x - 2) + (x^2 + 1)(3x - 2)'$$
$$= (2x)(3x - 2) + (x^2 + 1)(3)$$
$$= 6x^2 - 4x + 3x^2 + 3 = 9x^2 - 4x + 3$$

$$\boxed{9x^2 - 4x + 3}$$

**(c)** $h(x) = e^x \cos x$

$$h'(x) = (e^x)'\cos x + e^x(\cos x)'$$
$$= e^x \cos x + e^x(-\sin x)$$
$$= e^x(\cos x - \sin x)$$

$$\boxed{e^x(\cos x - \sin x)}$$

**(d)** $k(x) = x^2 \ln x$

$$k'(x) = (x^2)'\ln x + x^2(\ln x)' = 2x\ln x + x^2\left(\frac{1}{x}\right)$$
$$= 2x\ln x + x$$

$$\boxed{x(2\ln x + 1)}$$

### Problem 3 :

**Answer:**

**(a)** $f(x) = \frac{x+1}{x-1}$

$$f'(x) = \frac{()'(x-1) - (x+1)(x-1)'}{(x-1)^2}$$
$$= \frac{(1)(x-1) - (x+1)(1)}{(x-1)^2}$$
$$= \frac{x - 1 - x - 1}{(x-1)^2} = \frac{-2}{(x-1)^2}$$

$$\boxed{-\frac{2}{(x-1)^2}}$$

**(b)** $g(x) = \frac{\sin x}{x}$

$$g'(x) = \frac{(\sin x)'x - \sin x(x)'}{x^2} = \frac{x\cos x - \sin x}{x^2}$$

$$\boxed{\frac{x\cos x - \sin x}{x^2}}$$

**(c)** $h(x) = \frac{x^2-4}{x+2}$

Simplify first: $\frac{x^2-4}{x+2} = \frac{(x-2)(x+2)}{x+2} = x - 2$ for $x \neq -2$.
So $h'(x) = 1$ (for $x \neq -2$).

$$\boxed{1}$$

**(d)** $k(x) = \frac{e^x}{x^2+1}$

$$k'(x) = \frac{e^x(x^2 + 1) - e^x(2x)}{(x^2 + 1)^2} = \frac{e^x(x^2 - 2x + 1)}{(x^2 + 1)^2}$$
$$= \frac{e^x(x-1)^2}{(x^2 + 1)^2}$$

$$\boxed{\frac{e^x(x-1)^2}{(x^2 + 1)^2}}$$

### Problem 4 :

**Answer:**

**(a)** $f(x) = (2x^2 - 3)^5$

Inner function: $u = 2x^2 - 3$, $f = u^5$

$$f'(x) = 5u^4 \cdot u' = 5(2x^2 - 3)^4 \cdot 4x$$
$$= 20x(2x^2 - 3)^4$$

$$\boxed{20x(2x^2 - 3)^4}$$

**(b)** $g(x) = \sqrt{1 - x^2} = (1 - x^2)^{1/2}$

Inner: $u = 1 - x^2$, $g = u^{1/2}$

$$g'(x) = \frac{1}{2}u^{-1/2} \cdot u' = \frac{1}{2}(1 - x^2)^{-1/2} \cdot (-2x)$$
$$= -\frac{x}{\sqrt{1 - x^2}}$$

$$\boxed{-\frac{x}{\sqrt{1 - x^2}}}$$

**(c)** $h(x) = \cos(3x+1)$
Inner: $u = 3x+1$, $h = \cos u$

$$h'(x) = -\sin u \cdot u' = -\sin(3x+1)\cdot 3 = -3\sin(3x+1)$$

$$\boxed{-3\sin(3x+1)}$$

**(d)** $k(x) = e^{-x^2}$
Inner: $u = -x^2$, $k = e^u$

$$k'(x) = e^u \cdot u' = e^{-x^2}\cdot(-2x) = -2xe^{-x^2}$$

$$\boxed{-2xe^{-x^2}}$$

**(e)** $p(x) = \ln(x^2+x)$
Inner: $u = x^2+x$, $p = \ln u$

$$p'(x) = \frac{1}{u}\cdot u' = \frac{1}{x^2+x}\cdot(2x+1) = \frac{2x+1}{x^2+x}$$

$$\boxed{\frac{2x+1}{x^2+x}}$$

## Problem 5 :

**Answer:**

**(a)** $f(x) = x^3 - x$

$$f(1) = 1 - 1 = 0$$
$$f'(x) = 3x^2 - 1$$
$$f'(1) = 3(1) - 1 = 2$$

Tangent line: $y - f(1) = f'(1)(x-1) \Rightarrow y - 0 = 2(x-1) \Rightarrow y = 2x - 2$

$$\boxed{y = 2x - 2}$$

**(b)** $g(x) = \sqrt{x}$

$$g(4) = \sqrt{4} = 2$$
$$g'(x) = \frac{1}{2\sqrt{x}}$$
$$g'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

Tangent line: $y - 2 = \frac{1}{4}(x-4) \Rightarrow y = \frac{1}{4}x + 1$

$$\boxed{y = \frac{1}{4}x + 1}$$

**(c)** $h(x) = e^x$

$$h(0) = e^0 = 1$$
$$h'(x) = e^x$$
$$h'(0) = e^0 = 1$$

Tangent line: $y - 1 = 1(x-0) \Rightarrow y = x + 1$

$$\boxed{y = x + 1}$$

## Problem 6 :

**Answer:**

**(a)** $f(x) = x^3 - 12x + 5$
$f'(x) = 3x^2 - 12 = 3(x^2-4) = 3(x-2)(x+2)$
$f'(x) = 0 \Rightarrow x = 2$ or $x = -2$

$$\boxed{x = 2,\ -2}$$

**(b)** $g(x) = x^4 - 8x^2$

$$g'(x) = 4x^3 - 16x = 4x(x^2-4) = 4x(x-2)(x+2)$$

$g'(x) = 0 \Rightarrow x = 0,\ 2,\ -2$

$$\boxed{x = 0,\ 2,\ -2}$$

**(c)** $h(x) = \frac{x}{x^2+1}$

$$h'(x) = \frac{(1)(x^2+1) - x(2x)}{(x^2+1)^2} = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$$

$h'(x) = 0 \Rightarrow 1 - x^2 = 0 \Rightarrow x = \pm 1$

$$\boxed{x = 1,\ -1}$$

## Problem 7 :

**Answer:**

**(a)** $f(x) = x^2 - 6x + 8$

$$f'(x) = 2x - 6$$

At $x = 3$: $f'(3) = 0$
For $x < 3$, $f'(x) < 0$ (decreasing). For $x > 3$, $f'(x) > 0$ (increasing).
Sign change: $-$ to $+$ ⇒ **local minimum**.

$$f(3) = 9 - 18 + 8 = -1$$

$$\boxed{\text{Local min at } (3, -1)}$$

**(b)** $g(x) = -x^3 + 3x$

$$g'(x) = -3x^2 + 3 = 3(1 - x^2)$$

At $x = 1$: $g'(1) = 0$
For $x < 1$ (but $x > 0$), $g'(x) > 0$ (increasing).
For $x > 1$, $g'(x) < 0$ (decreasing).
Sign change: $+$ to $-$ ⇒ **local maximum**.

$$g(1) = -1 + 3 = 2$$

$$\boxed{\text{Local max at } (1,2)}$$

**(c)** $h(x) = x^4$

$$h'(x) = 4x^3$$

At $x = 0$: $h'(0) = 0$
For $x < 0$, $h'(x) < 0$ (decreasing). For $x > 0$, $h'(x) > 0$ (increasing).
Sign change: $-$ to $+$ ⇒ **local minimum**.

$$h(0) = 0$$

$$\boxed{\text{Local min at } (0,0)}$$

## Problem 8 :

**Answer:**

**(a)** $f(x) = 2x^3 - 9x^2 + 12x$

$$f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x-1)(x-2)$$

Critical points: $x = 1,\ 2$
Sign table:

| Interval | $x - 1$ | $x - 2$ | $f'(x)$ | Conclusion |
|---|---|---|---|---|
| $]-\infty, 1]$ | $-$ | $-$ | $+$ | Increasing |
| $[1,2]$ | $+$ | $-$ | $-$ | Decreasing |
| $[2,+\infty[$ | $+$ | $+$ | $+$ | Increasing |

$$\boxed{\text{Increasing on } ]-\infty,1] \cup [2,+\infty[\text{, Decreasing on } [1,2]}$$

**(b)** $g(x) = \frac{x}{x^2+4}$

$$g'(x) = \frac{(1)(x^2+4) - x(2x)}{(x^2+4)^2} = \frac{x^2+4-2x^2}{(x^2+4)^2}$$
$$= \frac{4-x^2}{(x^2+4)^2}$$

Critical points: $x = \pm 2$
Denominator always positive. Sign of numerator: $4 - x^2 > 0$ when $-2 < x < 2$.

| Interval | $4 - x^2$ | $g'(x)$ | Conclusion |
|---|---|---|---|
| $]-\infty,-2]$ | $-$ | $-$ | Decreasing |
| $[-2,2]$ | $+$ | $+$ | Increasing |
| $[2,+\infty[$ | $-$ | $-$ | Decreasing |

$\boxed{\text{Increasing on } [-2,2], \text{ Decreasing on } ]-\infty,-2] \cup [2,+\infty[}$

## Problem 9 :

### Answer:

**(a)** $f(x) = x^3 - 3x$ on $[-2,3]$

$$f'(x) = 3x^2 - 3 = 3(x-1)(x+1)$$

Critical points: $x = 1, \ -1$ (both in the interval).
Evaluate:

| $x$ | $f(x)$ |
|---|---|
| $-2$ | $-8+6=-2$ |
| $-1$ | $-1+3=2$ |
| $1$ | $1-3=-2$ |
| $3$ | $27-9=18$ |

Global max: $18$ at $x = 3$.
Global min: $-2$ at $x = -2$ and $x = 1$.

$\boxed{\text{Max } 18 \text{ at } x = 3, \text{Min } -2 \text{ at } x = -2 \text{ and } x = 1}$

**(b)** $g(x) = -x^2 + 4x$ on $[0,5]$

$$g'(x) = -2x + 4$$

Critical point: $x = 2$ (in the interval).
Evaluate:

| $x$ | $g(x)$ |
|---|---|
| $0$ | $0$ |
| $2$ | $-4+8=4$ |
| $5$ | $-25+20=-5$ |

Global max: $4$ at $x = 2$.
Global min: $-5$ at $x = 5$.

$\boxed{\text{Max } 4 \text{ at } x = 2, \text{Min } -5 \text{ at } x = 5}$

## Problem 10 :

### Answer:

**(a)** $f(x) = x^4 - 6x^2 + 1$

$$f'(x) = 4x^3 - 12x$$
$$f''(x) = 12x^2 - 12$$

$\boxed{f'(x) = 4x^3 - 12x, f''(x) = 12x^2 - 12}$

**(b)** $g(x) = \sin x + x$

$$g'(x) = \cos x + 1$$
$$g''(x) = -\sin x$$

$\boxed{g'(x) = \cos x + 1, g''(x) = -\sin x}$

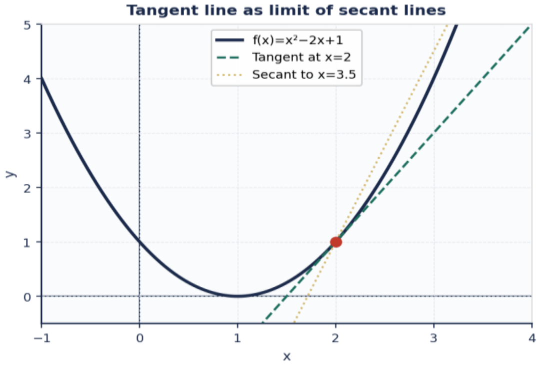
**(c)** $h(x) = xe^x$

$$h'(x) = (1)e^x + xe^x = e^x(1+x)$$
$$h''(x) = e^x(1+x) + e^x(1) = e^x(x+2)$$

$\boxed{h'(x) = e^x(1+x), h''(x) = e^x(x+2)}$

# Solution to Problems 11–35 — Medium

## Problem 11 :



### Answer:

**(a)** From the graph, the slope of the tangent at $x = 2$ is approximately 2.

**(b)** $f'(x) = 2(x-1)$

$$f'(2) = 2(2-1) = 2$$

Confirmed.

**(c)** $f(3.5) = (3.5-1)^2 = (2.5)^2 = 6.25$

$$f(2) = (2-1)^2 = 1$$

Slope of secant from $x = 2$ to $x = 3.5$:

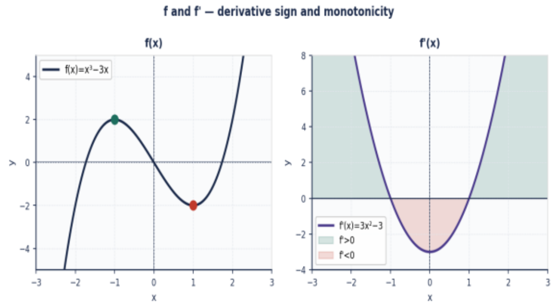
$$\frac{6.25-1}{3.5-2} = \frac{5.25}{1.5} = 3.5$$

**(d)** $(f(2+h) - f(2))/h = ((1+h)^2 - 1)/h = (1 + 2h + h^2 - 1)/h = 2 + h$

As $h \to 0$, this tends to 2, which is $f'(2)$. ✓

$$\boxed{\text{Tangent slope } = 2, \text{ secant slope } = 3.5}$$

## Problem 12 :



### Answer:

**(a)** $f'(x) = 0$ at $x = -1$ and $x = 1$.
These are the critical points of $f$.

**(b)** $f'(x) > 0$ on $(-\infty, -1)$ and $(1, \infty)$.
This means $f$ is **increasing** on those intervals.

**(c)** Local max at $x = -1$ (since $f'$ changes from $+$ to $-$).
Local min at $x = 1$ (since $f'$ changes from $-$ to $+$).

**(d)** Variation table:

$$\begin{array}{cccccccc} x & -\infty & & -1 & & 1 & & +\infty \\ f'(x) & & + & 0 & - & 0 & + & \\ f(x) & -\infty & \nearrow & 2 & \searrow & -2 & \nearrow & +\infty \end{array}$$

$$\boxed{\text{Inc on } ]-\infty, -1] \cup [1, +\infty[, \text{ Dec on } [-1, 1]}$$

## Problem 13 :

### Answer:

**(a)** $f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x-1)(x-3)$
Critical points: $x = 1, 3$

**(b)** Sign of $f'$:

- $f'(x) > 0$ on $(-\infty, 1)$ and $(3, \infty)$
- $f'(x) < 0$ on $(1, 3)$
  So $x = 1$: local max, $x = 3$: local min.

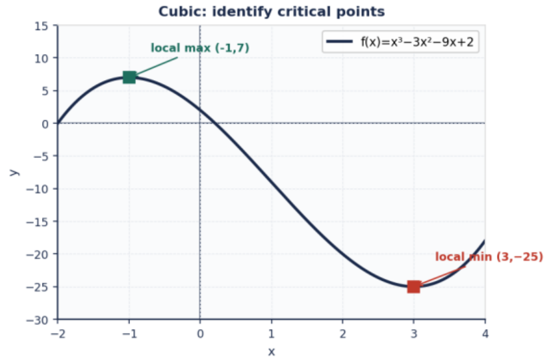
**(c)** Variation table:

$$\begin{array}{cccccccc} x & -\infty & & 1 & & 3 & & +\infty \\ f'(x) & & + & 0 & - & 0 & + & \\ f(x) & -\infty & \nearrow & 5 & \searrow & 1 & \nearrow & +\infty \end{array}$$

**(d)** $f(1) = 1 - 6 + 9 + 1 = 5$ (local max)
$f(3) = 27 - 54 + 27 + 1 = 1$ (local min)

$$\boxed{\text{Max } 5 \text{ at } x = 1, \text{ Min } 1 \text{ at } x = 3}$$

## Problem 14 :



### Answer:

**(a)** $f'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x+1)(x-3)$
Critical points: $x = -1, 3$

**(b)** $f(-1) = -1 - 3 + 9 + 2 = 7$ (local max)
$f(3) = 27 - 27 - 27 + 2 = -25$ (local min)

**(c)** Roots (from graph): approximately $x \approx -2.5, 0.2, 5.3$
$f(x) > 0$ on $(-2.5, 0.2) \cup (5.3, \infty)$

**(d)** Variation table:

$$\begin{array}{cccccccc} x & -\infty & & -1 & & 3 & & +\infty \\ f'(x) & & + & 0 & - & 0 & + & \\ f(x) & -\infty & \nearrow & 7 & \searrow & -25 & \nearrow & +\infty \end{array}$$

$$\boxed{\text{Inc on } (-\infty, -1] \cup [3, \infty), \text{ Dec on } [-1, 3]}$$

## Problem 15 :

### Answer:

**(a)** $f(x) = (x^2+1)^3 \cdot x^{1/2}$
Product rule:

$$f'(x) = 3(x^2+1)^2(2x) \cdot x^{1/2} + (x^2+1)^3 \cdot \frac{1}{2}x^{-1/2}$$

$$= 6x(x^2+1)^2\sqrt{x} + \frac{(x^2+1)^3}{2\sqrt{x}}$$

$$\boxed{6x(x^2+1)^2\sqrt{x} + \frac{(x^2+1)^3}{2\sqrt{x}}}$$

**(b)** $g(x) = \sin(x^2)\cos(x)$
Product rule:

$$g'(x) = \cos(x^2)(2x)\cos(x) + \sin(x^2)(-\sin(x))$$

$$\boxed{2x\cos(x^2)\cos(x) - \sin(x^2)\sin(x)}$$

**(c)** $h(x) = e^{\sqrt{x}}$
Chain rule: inner $u = \sqrt{x}$

$$h'(x) = e^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}$$

$$\boxed{\frac{e^{\sqrt{x}}}{2\sqrt{x}}}$$

**(d)** $k(x) = \frac{\ln(x^2+1)}{x}$
Quotient rule:

$$k'(x)=\frac{\frac{2x}{x^2+1}\cdot x-\ln(x^2+1)\cdot 1}{x^2}$$
$$=\frac{\frac{2x^2}{x^2+1}-\ln(x^2+1)}{x^2}$$
$$\boxed{\frac{2x^2-(x^2+1)\ln(x^2+1)}{x^2(x^2+1)}}$$

## Problem 16 :

### Answer:

**(a)** $f(-x)=(-x)^3+(-x)=-x^3-x=-(x^3+x)=-f(x)$ ✓

**(b)** $f'(x)=3x^2+1$
$f'(-x)=3(-x)^2+1=3x^2+1=f'(x)$ ✓

**(c)** General result: If $f$ is odd and differentiable, then $f'$ is even.
Proof: $f(-x)=-f(x)$. Differentiate both sides using the chain rule:
$-f'(-x)=-f'(x)\Rightarrow f'(-x)=f'(x)$. ✓

**(d)** Geometric interpretation: An odd function has rotational symmetry about the origin. Its tangent slope at $x$ equals the tangent slope at $-x$ (the slopes are mirrored).

$$\boxed{f' \text{ is even}}$$

## Problem 17 :

### Answer:

**(a)** $f'(x)=4x^3-12x^2+12x$
$$f''(x)=12x^2-24x+12=12(x^2-2x+1)$$
$$=12(x-1)^2$$
Since $(x-1)^2\geq 0$, $f''(x)\geq 0$ for all $x$.
The function is **concave up everywhere**. No inflection points.

**(b)** $g'(x)=3x^2-6$
$$g''(x)=6x$$
$g''(x)=0$ at $x=0$.
For $x<0$, $g''(x)<0$ → concave down.
For $x>0$, $g''(x)>0$ → concave up.
Inflection point at $x=0$: $g(0)=0\rightarrow(0,0)$.

$$\boxed{\text{(a) Always concave up, no inflection(b) Inflection at } (0,0)}$$

## Problem 18 :

### Answer:

**(a)** Critical points: $x=-1,\ 2,\ 4$
Sign of $f'$:

| Interval | $x+1$ | $x-2$ | $x-4$ | $f'(x)$ | Conclusion |
|---|---|---|---|---|---|
| $(-\infty,-$ | $-$ | $-$ | $-$ | $-$ | Decreasing |
| $(-1,2)$ | $+$ | $-$ | $-$ | $+$ | Increasing |
| $(2,4)$ | $+$ | $+$ | $-$ | $-$ | Decreasing |
| $(4,\infty)$ | $+$ | $+$ | $+$ | $+$ | Increasing |

Increasing on $(-1,2)$ and $(4,\infty)$.
Decreasing on $(-\infty,-1)$ and $(2,4)$.

**(b)** $x=-1$: $f'$ changes from $-$ to $+$ → **local min**
$x=2$: $f'$ changes from $+$ to $-$ → **local max**
$x=4$: $f'$ changes from $-$ to $+$ → **local min**

**(c)** 3 local extrema (2 minima, 1 maximum).

**(d)** No, we cannot determine the global max without knowing the actual function values.
Since $f'(x)>0$ for $x>4$, $f$ increases without bound as $x\to\infty$, so there is no global maximum.

$$\boxed{\text{(a) Inc on } (-1,2)\cup(4,\infty)\text{, Dec on } (-\infty,-1)\cup(2,4)}$$

## Problem 19 :

### Answer:

**(a)** Difference quotient: $\frac{(x+h)^3-x^3}{h}$

**(b)** Expand:
$$(x+h)^3=x^3+3x^2h+3xh^2+h^3$$
$$\frac{x^3+3x^2h+3xh^2+h^3-x^3}{h}=\frac{3x^2h+3xh^2+h^3}{h}$$
$$=3x^2+3xh+h^2$$

**(c)** As $h\to 0$, $3x^2+3xh+h^2\to 3x^2$.

$$\boxed{(x^3)'=3x^2}$$

**(d)** For general $n$:
$$(x+h)^n=x^n+nx^{n-1}h+\text{terms with } h^2$$
$$\frac{(x+h)^n-x^n}{h}=nx^{n-1}+\text{terms with } h\to nx^{n-1}$$
$$\boxed{(x^n)'=nx^{n-1}}$$

## Problem 20 :

### Answer:

**(a)** $f'(x)=\frac{(2x)(x^2+1)-(x^2-1)(2x)}{(x^2+1)^2}=\frac{2x^3+2x-2x^3+2x}{(x^2+1)^2}=\frac{4x}{(x^2+1)^2}$

**(b)** Denominator always positive.
$f'(x)>0$ for $x>0$, $f'(x)<0$ for $x<0$, $f'(x)=0$ at $x=0$.

**(c)** As $x\to\pm\infty$, $f(x)\to 1$. Horizontal asymptote: $y=1$.
Variation table:

| $x$ | $-\infty$ | | $0$ | | $+\infty$ |
|---|---|---|---|---|---|
| $f'(x)$ | | $-$ | $0$ | $+$ | |
| $f(x)$ | $1$ | $\searrow$ | $-1$ | $\nearrow$ | $1$ |

**(d)** Range: minimum value $-1$ at $x = 0$, approaches $1$ but never reaches it.

$$\boxed{[-1,1)}$$

## Problem 21 :

### Answer:

**(a)** Let one number be $x$. The other is $30 - x$.
$P(x) = x(30 - x) = 30x - x^2$, domain $0 < x < 30$.

**(b)** $P'(x) = 30 - 2x$
$P'(x) = 0 \Rightarrow x = 15$

**(c)** $P''(x) = -2 < 0 \rightarrow$ maximum.
Alternatively, $P'$ changes from $+$ to $-$ at $x = 15$.

**(d)** The two numbers are $15$ and $15$.
Maximum product: $15 \cdot 15 = 225$.

$$\boxed{15 \text{ and } 15, \text{ product } 225}$$

## Problem 22 :

### Answer:

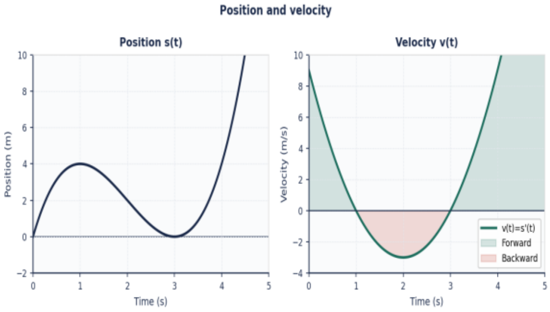
$$f'(x) = 3ax^2 + 2bx + c$$
$$f'(0) = c = 0$$
$$f(1) = a + b + 2 = 3 \Rightarrow a + b = 1$$
$$f'(1) = 3a + 2b = 0$$

Solve: $a + b = 1 \Rightarrow b = 1 - a$
$3a + 2(1 - a) = 0 \Rightarrow 3a + 2 - 2a = 0 \Rightarrow a = -2$
Then $b = 1 - (-2) = 3$

So $f(x) = -2x^3 + 3x^2 + 2$

$$\boxed{a = -2,\ b = 3,\ c = 0,\ d = 2}$$

## Problem 23 :



### Answer:

**(a)** $v(t) = 3t^2 - 12t + 9 = 3(t - 1)(t - 3)$
At rest: $v(t) = 0 \Rightarrow t = 1,\ 3$

**(b)** Forward (v > 0): $0 < t < 1$ and $t > 3$
Backward (v < 0): $1 < t < 3$

**(c)** $s(0) = 0$, $s(1) = 1 - 6 + 9 = 4$, $s(3) = 27 - 54 + 27 = 0$, $s(4) = 64 - 96 + 36 = 4$
Total distance: $|4 - 0| + |0 - 4| + |4 - 0| = 4 + 4 + 4 = 12$ m

**(d)** $a(t) = v'(t) = 6t - 12 = 6(t - 2)$
Decelerating when $v$ and $a$ have opposite signs:

- $0 < t < 2$: $v > 0,\ a < 0$
- $2 < t < 3$: $v < 0,\ a > 0$
  So decelerating on $(0, 2)$ and $(2, 3)$.

$$\boxed{\begin{array}{c}\text{(a) } t = 1{,}3 \text{(b) Forward on } (0,1) \cup (3,\infty) \text{(c) } 12 \text{ m} \\ \text{(d) Decelerating on } (0,2) \cup (2,3)\end{array}}$$

## Problem 24 :

### Answer:

**(a)** $\frac{f(4)-f(1)}{4-1} = \frac{16-1}{3} = 5$
$f'(x) = 2x \Rightarrow 2c = 5 \Rightarrow c = \frac{5}{2} = 2.5 \in (1,4)$

**(b)** $\frac{g(2)-g(0)}{2-0} = \frac{8-0}{2} = 4$
$g'(x) = 3x^2 \Rightarrow 3c^2 = 4 \Rightarrow c^2 = \frac{4}{3} \Rightarrow c = \frac{2}{\sqrt{3}} \approx 1.155 \in (0,2)$

**(c)** The tangent line at $x = c$ is parallel to the secant line joining the endpoints.

$$\boxed{\text{(a) } c = 2.5 \text{(b) } c = \frac{2}{\sqrt{3}}}$$

## Problem 25 :

### Answer:

**(a)** $f(x) = x^2(x^2 - 4x + 4) = x^2(x - 2)^2$
Zeros: $x = 0$ (double), $x = 2$ (double)

**(b)** $f'(x) = 4x^3 - 12x^2 + 8x = 4x(x^2 - 3x + 2) = 4x(x - 1)(x - 2)$
Critical points: $x = 0,\ 1,\ 2$
Sign of $f'$:

- $(-\infty, 0)$: $- \rightarrow$ decreasing
- $(0, 1)$: $+ \rightarrow$ increasing
- $(1, 2)$: $- \rightarrow$ decreasing
- $(2, \infty)$: $+ \rightarrow$ increasing
  So $x = 0$: local min ($f(0) = 0$), $x = 1$: local max ($f(1) = 1$), $x = 2$: local min ($f(2) = 0$)

**(c)** $f''(x) = 12x^2 - 24x + 8 = 4(3x^2 - 6x + 2)$
$f''(x) = 0 \Rightarrow 3x^2 - 6x + 2 = 0 \Rightarrow x = 1 \pm \frac{\sqrt{3}}{3}$
Two inflection points.

**(d)** As $x \to \pm\infty$, $f(x) \to +\infty$.
The graph touches the x-axis at $x = 0$ and $x = 2$, rises to a local max at $x = 1$, and goes to $+\infty$ at both ends.

$$\boxed{\begin{array}{c}\text{Zeros: } 0{,}2 \text{ (double); Local min at } 0{,}2;\ \text{Local max at } 1; \\ \text{Two inflections}\end{array}}$$

## Problem 26 :

### Answer:

**(a)** Marginal cost: $MC(x) = C'(x) = 0.03x^2 - 1.2x + 15$

**(b)** $MC'(x) = 0.06x - 1.2$
$MC'(x) = 0 \Rightarrow 0.06x = 1.2 \Rightarrow x = 20$

**(c)** $MC''(x) = 0.06 > 0 \rightarrow$ minimum

$$MC(20) = 0.03(400) - 1.2(20) + 15$$
$$= 12 - 24 + 15 = 3$$

$$\boxed{x = 20, \text{ minimum marginal cost } = 3}$$

## Problem 27 :

**Answer:**

**(a)** Define $g(x) = \frac{f(x)}{e^x}$
$g'(x) = \frac{f'(x)e^x - f(x)e^x}{e^{2x}} = \frac{(f'(x)-f(x))e^x}{e^{2x}} =$
0 since $f'(x) = f(x)$.

**(b)** $g'(x) = 0$ for all $x$, so $g(x)$ is constant.

**(c)** $g(0) = \frac{f(0)}{e^0} = \frac{1}{1} = 1$
So $g(x) = 1$ for all $x$, meaning $\frac{f(x)}{e^x} = 1 \Rightarrow f(x) = e^x$.

$$\boxed{f(x) = e^x}$$

## .Problem 28 :

**Answer:**

**(a)** The cylinder of height $h$ has radius $r^2 = R^2 - \left(\frac{h}{2}\right)^2$

$$V(h) = \pi r^2 h = \pi\left(R^2 - \frac{h^2}{4}\right)h = \pi\left(R^2 h - \frac{h^3}{4}\right)$$

**(b)** $V'(h) = \pi\left(R^2 - \frac{3h^2}{4}\right) = 0 \Rightarrow R^2 = \frac{3h^2}{4} \Rightarrow h = \frac{2R}{\sqrt{3}}$

**(c)** $r^2 = R^2 - \frac{h^2}{4} = R^2 - \frac{R^2}{3} = \frac{2R^2}{3}$

$$V_{\max} = \pi\left(\frac{2R^2}{3}\right)\left(\frac{2R}{\sqrt{3}}\right) = \frac{4\pi R^3}{3\sqrt{3}}$$

$$\boxed{V_{\max} = \frac{4\pi R^3}{3\sqrt{3}}}$$

## Problem 29 :

**Answer:**

**(a)** $f'(x) = 1 - \cos(x) \geq 0$ since $\cos(x) \leq 1$ for all $x$.

**(b)** $f'(x) \geq 0$, so $f$ is non-decreasing on $[0, \infty)$.
$$f(0) = 0 - \sin(0) = 0$$
For $x \geq 0$, $f(x) \geq f(0) = 0$, so $x - \sin(x) \geq 0$.

**(c)** Therefore, $\sin(x) \leq x$ for all $x \geq 0$.

$$\boxed{\sin(x) \leq x \text{ for } x \geq 0}$$

## Problem 30 :

**Answer:** **(a)** Continuity at $x = 1$:
$$\lim_{x\to 1^-} f(x) = a(1)^2 + 3 = a + 3$$
$$\lim_{x\to 1^+} f(x) = b(1) + 1 = b + 1$$
For continuity: $a + 3 = b + 1 \Rightarrow a - b = -2$

**(b)** Differentiability at $x = 1$:
Left derivative: $f'(x) = 2ax$ for $x < 1$,
so $f'(1^-) = 2a$
Right derivative: $f'(x) = b$ for $x > 1$, so $f'(1^+) = b$
For differentiability: $2a = b$

**(c)** Solve: $b = 2a$
Substitute into $a - b = -2$: $a - 2a = -2 \Rightarrow -a = -2 \Rightarrow a = 2$
Then $b = 4$

$$\boxed{a = 2,\ b = 4}$$

## Problem 31 :

**Answer:**

**(a)** $f'(x) = e^{-x}(1 - x)$
Critical point: $x = 1$
$f'(x) > 0$ for $x < 1$, $f'(x) < 0$ for $x > 1$ → local max at $x = 1$
$$f(1) = e^{-1} = \frac{1}{e}$$

**(b)** $\lim_{x\to+\infty} xe^{-x} = 0$
$$\lim_{x\to-\infty} xe^{-x} = -\infty$$

**(c)** Variation table:

| | | | | | |
|---|---|---|---|---|---|
| $x$ | $-\infty$ | | $1$ | | $+\infty$ |
| $f'(x)$ | | $+$ | $0$ | $-$ | |
| $f(x)$ | $-\infty$ | $\nearrow$ | $\frac{1}{e}$ | $\searrow$ | $0$ |

Horizontal asymptote: $y = 0$ as $x \to +\infty$.

**(d)** $f''(x) = e^{-x}(x - 2)$
$f''(x) = 0$ at $x = 2$.
For $x < 2$, $f''(x) < 0$ → concave down.
For $x > 2$, $f''(x) > 0$ → concave up.
Inflection point at $x = 2$: $f(2) = 2e^{-2}$.

$$\boxed{\text{Max at } (1, 1/e), \text{ Inflection at } (2, 2/e^2)}$$

## Problem 32 :

**Answer:**

**(a)** $f(1) = 1 + 1 - 5 = -3 < 0$
$$f(2) = 8 + 2 - 5 = 5 > 0$$

**(b)** $f$ is continuous, so by the Intermediate Value Theorem, there is at least one root in $(1, 2)$.

**(c)** $f'(x) = 3x^2 + 1 > 0$ for all $x$, so $f$ is strictly increasing.
A strictly increasing function can have at most one zero.

**(d)** Therefore, there is exactly one root in $(1, 2)$.

$$\boxed{\text{Exactly one root in } (1,2)}$$

## Problem 33 :

**Answer:**

**(a)** The tangent at $(a, f(a))$ has slope $f'(a) = 3a^2 - 6a$.
Equation: $y - f(a) = f'(a)(x - a)$

**(b)** For the tangent to pass through $(0, 0)$:
$0 - f(a) = f'(a)(0 - a) \Rightarrow -f(a) = -af'(a) \Rightarrow f(a) = af'(a)$

**(c)** $f(a) = a^3 - 3a^2$, $f'(a) = 3a^2 - 6a$
$$a^3 - 3a^2 = a(3a^2 - 6a) = 3a^3 - 6a^2$$
$a^3 - 3a^2 - 3a^3 + 6a^2 = 0 \Rightarrow -2a^3 + 3a^2 = 0 \Rightarrow a^2(-2a + 3) = 0$
$a = 0$ or $a = \frac{3}{2}$

**(d)** Points: $(0,0)$ and $\left(\frac{3}{2}, \frac{27}{8} - \frac{27}{4}\right) = \left(\frac{3}{2}, -\frac{27}{8}\right)$

$$\boxed{(0,0) \text{ and } \left(\frac{3}{2}, -\frac{27}{8}\right)}$$

## Problem 34 :

### Answer:

**(a)** $a(t) = v'(t) = 20 - 10t$

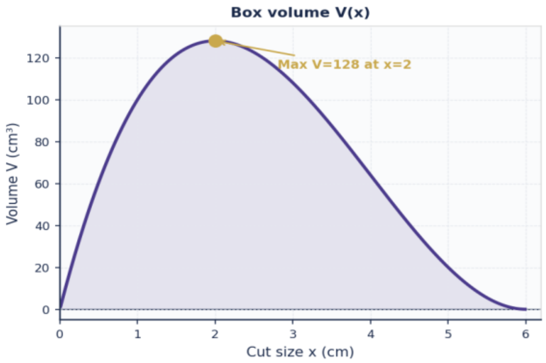
**(b)** $a(t) > 0$ when $t < 2 \rightarrow$ accelerating on $(0,2)$
$a(t) < 0$ when $t > 2 \rightarrow$ decelerating on $(2,5]$

**(c)** $v'(t) = 0 \Rightarrow 20 - 10t = 0 \Rightarrow t = 2$
$v(2) = 60 + 40 - 20 = 80$ km/h

**(d)** $v(t) = 60 \Rightarrow 60 + 20t - 5t^2 = 60 \Rightarrow 20t - 5t^2 = 0 \Rightarrow 5t(4 - t) = 0 \Rightarrow t = 0$ or $t = 4$
$t = 4$ hours.

$$\boxed{\text{(a) } a(t) = 20 - 10t\text{(b) Accelerating on } (0,2), \text{ Decelerating on } (2,5]\text{(c) } 80 \text{ km/h at } t = 2\text{(d) } t = 4}$$

## Problem 35 :



### Answer:

**(a)** From the graph, maximum volume is approximately $128$ at $x \approx 2$.

**(b)** $V(x) = x(12 - 2x)^2 = x(144 - 48x + 4x^2) = 144x - 48x^2 + 4x^3$

$$V'(x) = 144 - 96x + 12x^2 = 12(x^2 - 8x + 12)$$
$$= 12(x - 2)(x - 6)$$

$V'(x) = 0 \Rightarrow x = 2$ or $x = 6$

**(c)** Domain: $0 < x < 6$ (since $12 - 2x > 0$).
So $x = 2$ is the critical point in the domain.

$$V(2) = 2(12 - 4)^2 = 2(8)^2 = 128$$

$V''(x) = -96 + 24x$, $V''(2) = -96 + 48 = -48 < 0 \rightarrow$ maximum.

**(d)** Domain is $(0,6)$ because we need $12 - 2x > 0$ for a positive base length.

$$\boxed{\text{Max volume } 128 \text{ at } x = 2}$$

# Solutions to Problems 36–50 — Difficult & Real-World

## Problem 36 :

### Answer:

**(a)** Difference quotient for $fg$ at $x$:

$$\frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

**(b)** Add and subtract $f(x+h)g(x)$:

$$\frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

**(c)** Split into two fractions:

$$f(x+h)\frac{g(x+h) - g(x)}{h} + g(x)\frac{f(x+h) - f(x)}{h}$$

As $h \rightarrow 0$:
$f(x+h) \rightarrow f(x)$ (continuity of $f$)

$$\frac{g(x+h) - g(x)}{h} \rightarrow g'(x)$$
$$\frac{f(x+h) - f(x)}{h} \rightarrow f'(x)$$

So the limit is $f(x)g'(x) + g(x)f'(x)$. ✓

$$\boxed{(fg)' = f'g + fg'}$$

## Problem 37 :

### Answer:

**(a)** $x^2 + y^2 = 100$ (by Pythagoras)

**(b)** Differentiate with respect to $t$:

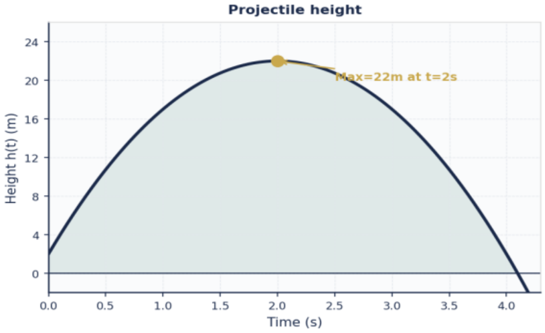
$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

**(c)** $\frac{dx}{dt} = 2$ m/s, $x = 6$, $y = \sqrt{100 - 36} = 8$
$2(6)(2) + 2(8)\frac{dy}{dt} = 0 \Rightarrow 24 + 16\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = -\frac{3}{2}$ m/s

**(d)** $\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt} = -\frac{2x}{\sqrt{100-x^2}}$
The speed of the top is $|\frac{dy}{dt}| = \frac{2x}{\sqrt{100-x^2}}$.
This increases as $x$ increases toward 10.
The fastest occurs as $x \rightarrow 10^-$.

$$\boxed{\text{(c) } -\frac{3}{2} \text{ m/s(d) } x \rightarrow 10^-}$$

## Problem 38 :



### Answer:

**(a)** $h'(t) = -10t + 20 = 0$ at $t = 2$
$h'(t) > 0$ for $t < 2 \rightarrow$ rising
$h'(t) < 0$ for $t > 2 \rightarrow$ falling
Rising on $(0,2)$, falling on $(2, t_{\text{land}})$

**(b)** Maximum height: $h(2) = -5(4) + 20(2) + 2 = -20 + 40 + 2 = 22$ m
Reached at $t = 2$ s

**(c)** $h(t) = 0 \Rightarrow -5t^2 + 20t + 2 = 0$
$5t^2 - 20t - 2 = 0$

$$t = \frac{20 \pm \sqrt{400 + 40}}{10} = \frac{20 \pm \sqrt{440}}{10} = \frac{20 \pm 2\sqrt{110}}{10}$$
$$= 2 \pm \frac{\sqrt{110}}{5}$$

Positive root: $t = 2 + \frac{\sqrt{110}}{5} \approx 4.10$ s

**(d)** Speed at impact: $| h'(t_{\text{land}}) | = | -10t + 20 |$

$$h'(t_{\text{land}}) = -10\left(2 + \frac{\sqrt{110}}{5}\right) + 20$$
$$= -20 - 2\sqrt{110} + 20 = -2\sqrt{110}$$

Speed $= 2\sqrt{110} \approx 20.98$ m/s

$$\boxed{\begin{array}{c}\text{(a) Rising on } (0,2)\text{, falling on } (2,4.10) \\ \text{(b) 22 m at } t = 2 \text{(c) } t \approx 4.10 \text{ s(d) } 2\sqrt{110} \text{ m/s}\end{array}}$$

## Problem 39 :

### Answer:

**(a)** If $f$ is constant on $[a, b]$, then $f'(x) = 0$ for all $x \in (a, b)$. Done.

**(b)** If $f$ is not constant, since it is continuous on a closed interval, it attains a maximum and a minimum. Since $f(a) = f(b)$ and the function is not constant, at least one extremum occurs at an interior point $c \in (a, b)$.

**(c)** At an interior extremum $c$, $f'(c)$ must equal 0 (if $f'(c) > 0$, the function would be increasing through $c$, contradicting a max/min; if $f'(c) < 0$, similarly). ✓

$$\boxed{\text{Rolle's Theorem proved}}$$

## Problem 40 :

### Answer:

**(a)** $y(0) = 0$ ✓, $y(4) = 16(0) = 0$ ✓

$$y'(x) = 2x(4-x)^2 + x^2(2)(4-x)(-1)$$
$$= 2x(4-x)[(4-x) - x]$$
$$= 2x(4-x)(4-2x)$$

$y'(0) = 0$ ✓, $y'(4) = 0$ ✓

**(b)** $y'(x) = 0 \Rightarrow x = 0,\ 4,\ 2$
Interior point: $x = 2$

**(c)** $y(2) = 4(2)^2 = 4(4) = 16$ units

**(d)** 16 > 15 → **Not satisfied**

$$\boxed{\text{Max deflection 16 units, specification not met}}$$

## Problem 41 :

### Answer:

**(a)** $P(q) = R(q) - C(q) = (40q - 0.5q^2) - (q^3 - 6q^2 + 20q + 10)$
$= -q^3 + 5.5q^2 + 20q - 10$

**(b)** $P'(q) = -3q^2 + 11q + 20$
$-3q^2 + 11q + 20 = 0 \Rightarrow 3q^2 - 11q - 20 = 0$
$\Delta = 121 + 240 = 361 = 19^2$
$q = \frac{11 \pm 19}{6} \Rightarrow q = 5$ or $q = -\frac{4}{3}$ (invalid)

**(c)** $P''(q) = -6q + 11$, $P''(5) = -30 + 11 = -19 < 0 \rightarrow$ maximum
$P(5) = -125 + 5.5(25) + 100 - 10$
$= -125 + 137.5 + 90 = 102.5$

**(d)** $MR = R'(q) = 40 - q$, $MC = C'(q) = 3q^2 - 12q + 20$
At $q = 5$: $MR = 35$, $MC = 75 - 60 + 20 = 35$ ✓

$$\boxed{q = 5\text{, max profit 102.5 thousand DA}}$$

## Problem 42 :

### Answer: **(a)** $\lim_{x \to 0} \frac{\sin x}{x}$ (form $0/0$)

L'Hôpital: $\lim_{x \to 0} \frac{\cos x}{1} = \cos 0 = 1$

$$\boxed{1}$$

**(b)** $\lim_{x \to \infty} \frac{x^2}{e^x}$ (form $\infty/\infty$)
First L'Hôpital: $\lim_{x \to \infty} \frac{2x}{e^x}$ (still $\infty/\infty$)
Second L'Hôpital: $\lim_{x \to \infty} \frac{2}{e^x} = 0$

$$\boxed{0}$$

**(c)** $\lim_{x \to 1} \frac{x^3 - 1}{x - 1}$ (form $0/0$)
L'Hôpital: $\lim_{x \to 1} \frac{3x^2}{1} = 3$

$$\boxed{3}$$

**(d)** $\lim_{x \to 0^+} x \ln x$ (form $0 \cdot (-\infty)$)
Rewrite as $\frac{\ln x}{1/x}$ (form $-\infty/\infty$)
L'Hôpital: $\lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} (-x) = 0$

$$\boxed{0}$$

## Problem 43 :

### Answer:

**(a)** $N'(t) = 1000(0.3)e^{0.3t} = 300e^{0.3t}$

**(b)** $0.3N(t) = 0.3(1000e^{0.3t}) = 300e^{0.3t} = N'(t)$ ✓

**(c)** $N'(t) = 1000 \Rightarrow 300e^{0.3t} = 1000 \Rightarrow e^{0.3t} = \frac{10}{3}$
$0.3t = \ln\left(\frac{10}{3}\right) \Rightarrow t = \frac{\ln(10/3)}{0.3} \approx 4.01$ hours

**(d)** Doubling time: $N(t) = 2000 \Rightarrow 1000e^{0.3t} = 2000 \Rightarrow e^{0.3t} = 2$
$0.3t = \ln 2 \Rightarrow t = \frac{\ln 2}{0.3} \approx 2.31$ hours

$$\boxed{\text{(a) } 300e^{0.3t} \text{(c) 4.01 h(d) 2.31 h}}$$

## Problem 44 :

**Answer:**

**(a)** $x'(t) = -\sin t,\ y'(t) = \cos t$

**(b)** Speed $v(t) = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1$

**(c)** Position: $(x, y) = (\cos t, \sin t)$,
Velocity: $(-\sin t, \cos t)$
Dot product: $\cos t(-\sin t) + \sin t(\cos t) = -\sin t\cos t + \sin t\cos t = 0$ ✓

**(d)** For ellipse: $x = 2\cos t,\ y = \sin t$

$$x'(t) = -2\sin t,\ y'(t) = \cos t$$

$$v(t) = \sqrt{4\sin^2 t + \cos^2 t} = \sqrt{1 + 3\sin^2 t}$$

Max when $\sin^2 t = 1 \Rightarrow v = \sqrt{1+3} = 2$ (at $t = \pi/2, 3\pi/2$)
Min when $\sin^2 t = 0 \Rightarrow v = 1$ (at $t = 0, \pi$)

$$\boxed{\text{Max speed 2, Min speed 1}}$$

## Problem 45 :

**Answer:**

**(a)** $\lim\limits_{h\to 0} \frac{f(h)-f(0)}{h} = \lim\limits_{h\to 0} \frac{h^{1/3}}{h} = \lim\limits_{h\to 0} h^{-2/3}$

**(b)** As $h \to 0$, $h^{-2/3} \to +\infty$ (since the exponent is negative).
The limit does not exist as a finite number, so $f'(0)$ does not exist.

**(c)** $\lim\limits_{x\to 0} x^{1/3} = 0 = f(0) \to$ continuous.

**(d)** The graph has a vertical tangent at $x = 0$ (the tangent line is vertical).

$$\boxed{f \text{ is continuous but not differentiable at } 0}$$

## Problem 46 :

**Answer:**

**(a)** $C'(t) = 20e^{-0.5t}(1 - 0.5t) = 0 \Rightarrow t = 2$ hours

**(b)** $C(2) = 20(2)e^{-1} = 40e^{-1} \approx 14.7$ mg/L

**(c)** $20te^{-0.5t} = 5 \Rightarrow te^{-0.5t} = 0.25$
Numerically: $t_1 \approx 0.27$ h, $t_2 \approx 7.64$ h
Duration: $t_2 - t_1 \approx 7.37$ h

**(d)** $20te^{-0.5t} = 8 \Rightarrow te^{-0.5t} = 0.4$
Numerical solution after peak: $t \approx 4.15$ hours

$$\boxed{\text{(a) } t = 2 \text{ h(b) 14.7 mg/L(c) 7.37 h(d) 4.15 h}}$$

## Problem 47 :

**Answer:** (a) $f'(x) = \cos x + \cos 3x$.
Using $\cos 3x = 4\cos^3 x - 3\cos x$:
$f'(x) = \cos x + 4\cos^3 x - 3\cos x = 4\cos^3 x - 2\cos x = 2\cos x(2\cos^2 x - 1)$.
Factorised form: $f'(x) = 2\cos x(2\cos^2 x - 1)$.

(b) $f'(x) = 0 \Rightarrow \cos x = 0$ or $\cos^2 x = \frac{1}{2}$ (i.e. $\cos x = \pm\frac{\sqrt{2}}{2}$).
In $[0, 2\pi]$:

- $\cos x = 0 \Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$
- $\cos x = \frac{\sqrt{2}}{2} \Rightarrow x = \frac{\pi}{4}, \frac{7\pi}{4}$
- $\cos x = -\frac{\sqrt{2}}{2} \Rightarrow x = \frac{3\pi}{4}, \frac{5\pi}{4}$
  Critical points: $x = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}$.

(c) Evaluate $f$ at these points and at the endpoints $0, 2\pi$:

$$f(0) = 0$$

$$f(\pi/4) = \frac{\sqrt{2}}{2} + \frac{1}{3}\cdot\frac{\sqrt{2}}{2} = \frac{2\sqrt{2}}{3} \approx 0.9428$$

$$f(\pi/2) = 1 + \frac{1}{3}(-1) = \frac{2}{3} \approx 0.6667$$

$$f(3\pi/4) = \frac{\sqrt{2}}{2} + \frac{1}{3}\left(-\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{3} \approx 0.4714$$

$$f(5\pi/4) = -\frac{\sqrt{2}}{2} + \frac{1}{3}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{2\sqrt{2}}{3} \approx -0.9428$$

$$f(3\pi/2) = -1 + \frac{1}{3}(1) = -\frac{2}{3} \approx -0.6667$$

$$f(7\pi/4) = -\frac{\sqrt{2}}{2} + \frac{1}{3}\cdot\frac{\sqrt{2}}{2} = -\frac{\sqrt{2}}{3} \approx -0.4714$$

$$f(2\pi) = 0$$

The largest value is $\frac{2\sqrt{2}}{3}$.

Global maximum: $\boxed{\frac{2\sqrt{2}}{3}}$

(d) Attained at $x = \frac{\pi}{4}$ (and also at $x = \frac{\pi}{4} + 2\pi$ but only $[0, 2\pi]$ considered).
Answer: $\boxed{x = \frac{\pi}{4}}$

## Problem 48 :

**Answer:**

**(a)** $f(x) = x^2 - 2$, $f'(x) = 2x$
$x_0 = 1$

$$x_1 = 1 - \frac{1-2}{2} = 1 - \frac{-1}{2} = 1.5$$

$$x_2 = 1.5 - \frac{2.25-2}{3} = 1.5 - \frac{0.25}{3} = 1.5 - 0.08333$$
$$= 1.41667$$

$$x_3 = 1.41667 - \frac{(1.41667)^2 - 2}{2(1.41667)}$$
$$\approx 1.41667 - \frac{2.00694 - 2}{2.83334}$$
$$= 1.41667 - 0.00245 = 1.41422$$

$$x_4 \approx 1.41422 - \frac{(1.41422)^2 - 2}{2(1.41422)}$$
$$\approx 1.41422 - \frac{2.00002 - 2}{2.82844}$$
$$= 1.41421$$

**(b)** Correct decimal places:
$x_0$: 0 correct, $x_1$: 1, $x_2$: 3, $x_3$: 5, $x_4$: 7 (approximately).
Convergence is quadratic.

**(c)** Each iteration draws the tangent line at $(x_n, f(x_n))$ and finds where it crosses the x-axis. That point becomes the next approximation.

**(d)** If $f'(x_n) = 0$, the tangent is horizontal and never crosses the x-axis. The method fails (division by zero).

$$\boxed{x_4 \approx 1.41421}$$

## Problem 49 :

### Answer:

**(a)** Let $f(x) = x + \frac{a}{x}$ for $x > 0$ (where $a > 0$ is constant).

$$f'(x) = 1 - \frac{a}{x^2}$$

$f'(x) = 0 \Rightarrow x^2 = a \Rightarrow x = \sqrt{a}$ (positive root)

**(b)** $f''(x) = \frac{2a}{x^3} > 0$ for $x > 0$, so $x = \sqrt{a}$ gives a minimum.
Minimum value: $f(\sqrt{a}) = \sqrt{a} + \frac{a}{\sqrt{a}} = 2\sqrt{a}$

**(c)** Let $x = \sqrt{ab}$? Actually, set $x = \sqrt{b/a}$...
Simpler:
Take $a$ and $b$ directly:
$(\sqrt{a} - \sqrt{b})^2 \geq 0 \Rightarrow a - 2\sqrt{ab} + b \geq 0 \Rightarrow \frac{a+b}{2} \geq \sqrt{ab}$

**(d)** Equality occurs when $\sqrt{a} = \sqrt{b}$, i.e., $a = b$.

$$\boxed{\text{AM-GM proved}}$$

## Problem 50 :

### Answer:

**(a)** Time in air: $\frac{\sqrt{x^2+d_1^2}}{c}$

Time in water: $\frac{\sqrt{(L-x)^2+d_2^2}}{c/n} = \frac{n\sqrt{(L-x)^2+d_2^2}}{c}$

So $T(x) = \frac{\sqrt{x^2+d_1^2}}{c} + \frac{n\sqrt{(L-x)^2+d_2^2}}{c}$

**(b)** $T'(x) = \frac{x}{c\sqrt{x^2+d_1^2}} - \frac{n(L-x)}{c\sqrt{(L-x)^2+d_2^2}} = 0$

So $\frac{x}{\sqrt{x^2+d_1^2}} = \frac{n(L-x)}{\sqrt{(L-x)^2+d_2^2}}$

**(c)** $\sin\theta_1 = \frac{x}{\sqrt{x^2+d_1^2}}$, $\sin\theta_2 = \frac{L-x}{\sqrt{(L-x)^2+d_2^2}}$

The equation becomes $\sin\theta_1 = n\sin\theta_2$
So $\frac{\sin\theta_1}{\sin\theta_2} = n \Rightarrow \frac{\sin\theta_1}{c} = \frac{\sin\theta_2}{c/n}$ (Snell's law).

**(d)** Solve $\frac{x}{\sqrt{x^2+9}} = 1.33 \cdot \frac{5-x}{\sqrt{(5-x)^2+16}}$
Numerical solution: $x \approx 2.1$ m.

$$\boxed{x \approx 2.1\text{ m}}$$

.

### CHAPTER 5 — SEQUENCES

# SOLUTIONS OF PROBLEMS

## Problems 1–15 — Instant Recall

### Problem 1 :

**Answer:**

**(a)** $u_n = 4n + 1$

$u_1 = 4(1) + 1 = 5,\ u_2 = 9,\ u_3 = 13,\ u_4 = 17,\ u_5 = 21$

Common difference: $u_{n+1} - u_n = 4$ → **Arithmetic, d = 4**

$$\boxed{5,\ 9,\ 13,\ 17,\ 21\ \text{(arithmetic, } d = 4)}$$

**(b)** $u_n = 3 \cdot 2^n$

$u_1 = 3(2) = 6,\ u_2 = 12,\ u_3 = 24,\ u_4 = 48,\ u_5 = 96$

Common ratio: $u_{n+1}/u_n = 2$ → **Geometric, q = 2**

$$\boxed{6,\ 12,\ 24,\ 48,\ 96\ \text{(geometric, } q = 2)}$$

**(c)** $u_0 = 5,\ u_{n+1} = u_n + 7$

$u_0 = 5,\ u_1 = 12,\ u_2 = 19,\ u_3 = 26,\ u_4 = 33$

Common difference: $u_{n+1} - u_n = 7$ → **Arithmetic, d = 7**

$$\boxed{5,\ 12,\ 19,\ 26,\ 33\ \text{(arithmetic, } d = 7)}$$

**(d)** $u_n = n^2 + 2$

$u_1 = 1 + 2 = 3,\ u_2 = 4 + 2 = 6,\ u_3 = 9 + 2 = 11,\ u_4 = 16 + 2 = 18,\ u_5 = 25 + 2 = 27$

No common difference or ratio → **Other (quadratic)**

$$\boxed{3,\ 6,\ 11,\ 18,\ 27\ \text{(other)}}$$

### Problem 2 :

**Answer:**

**(a)** $2,5,8,11,14$

$5 - 2 = 3,\ 8 - 5 = 3$ → **Arithmetic, d = 3**

$$\boxed{\text{Arithmetic, } d = 3}$$

**(b)** $3,6,12,24$

$6/3 = 2,\ 12/6 = 2$ → **Geometric, q = 2**

$$\boxed{\text{Geometric, } q = 2}$$

**(c)** $1,4,9,16$

These are perfect squares: $1^2, 2^2, 3^2, 4^2$. No common difference or ratio.

$$\boxed{\text{Neither}}$$

**(d)** $100,10,1,0.1$

$10/100 = 1/10,\ 1/10 = 1/10,\ 0.1/1 = 1/10$ → **Geometric, q = 1/10**

$$\boxed{\text{Geometric, } q = \frac{1}{10}}$$

### Problem 3 :

**Answer:**

**(a)** $u_1 = 5$

$$u_{10} = 5 + 4(9) = 5 + 36 = 41$$
$$u_{20} = 5 + 4(19) = 5 + 76 = 81$$
$$\boxed{u_1 = 5,\ u_5 = 21,\ u_{10} = 41,\ u_{20} = 81}$$

**(b)** $u_n = 5 + 4(n - 1) = 5 + 4n - 4 = 4n + 1$

$$\boxed{u_n = 4n + 1}$$

**(c)** $4n + 1 = 89 \Rightarrow 4n = 88 \Rightarrow n = 22$

$$\boxed{n = 22}$$

### Problem 4 :

**Answer: :**

**(a)** $u_1 = 2$

$$u_n = u_1 q^{n-1}$$
$$u_4 = 2 \cdot 3^3 = 2 \cdot 27 = 54$$
$$u_6 = 2 \cdot 3^5 = 2 \cdot 243 = 486$$
$$\boxed{u_1 = 2,\ u_4 = 54,\ u_6 = 486}$$

**(b)** $u_n = 2 \cdot 3^{n-1}$

$$\boxed{u_n = 2 \cdot 3^{n-1}}$$

**(c)** $2 \cdot 3^{n-1} > 1000 \Rightarrow 3^{n-1} > 500$

$3^5 = 243,\ 3^6 = 729$

So $n - 1 = 6 \Rightarrow n = 7$

$$\boxed{n = 7}$$

### Problem 5 :

**Answer: (a)** $S_n = \frac{n(u_1 + u_n)}{2}$

$n = 100,\ u_1 = 1,\ u_{100} = 100$

$$S = \frac{100(1 + 100)}{2} = 50(101) = 5050$$
$$\boxed{5050}$$

**(b)** First find $n$: $u_n = u_1 + (n - 1)d$

$50 = 5 + (n - 1)3 \Rightarrow 50 - 5 = 3(n - 1) \Rightarrow 45 = 3(n - 1) \Rightarrow n - 1 = 15 \Rightarrow n = 16$

$$S = \frac{16(5 + 50)}{2} = 8(55) = 440$$
$$\boxed{440}$$

**(c)** $u_{20} = 3 + 19(5) = 3 + 95 = 98$

$$S_{20} = \frac{20(3 + 98)}{2} = 10(101) = 1010$$
$$\boxed{1010}$$

### Problem 6 :

**Answer:**

**(a)** $1 + 2 + 4 + \cdots + 256$

$u_1 = 1,\ q = 2.\ 256 = 2^8$ so $n = 9$ (since $2^0 = 1, 2^8 = 256$).

$$S_n = \frac{u_1(q^n - 1)}{q - 1} = \frac{1(2^9 - 1)}{2 - 1} = 512 - 1 = 511$$
$$\boxed{511}$$

**(b)** $u_1 = 3,\ q = 2,\ n = 6$
$$S_6 = \frac{3(2^6 - 1)}{2 - 1} = 3(64 - 1) = 3(63) = 189$$
$$\boxed{189}$$

**(c)** $3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \frac{3}{16}$

$u_1 = 3,\ q = \frac{1}{2},\ n = 5$
$$S_5 = \frac{3\left(1 - \left(\frac{1}{2}\right)^5\right)}{1 - \frac{1}{2}} = 3 \cdot \frac{1 - \frac{1}{32}}{\frac{1}{2}} = 3 \cdot \frac{\frac{31}{32}}{\frac{1}{2}} = 3 \cdot \frac{31}{16}$$
$$= \frac{93}{16}$$
$$\boxed{\frac{93}{16}}$$

## Problem 7 :

### Answer:

**(a)** $u_0 = 1$
$$u_1 = 2(1) + 3 = 5$$
$$u_2 = 2(5) + 3 = 13$$
$$u_3 = 2(13) + 3 = 29$$
$$u_4 = 2(29) + 3 = 61$$
$$\boxed{1,\ 5,\ 13,\ 29,\ 61}$$

**(b)** $u_1 = 64$
$$u_2 = \frac{64}{2} = 32$$
$$u_3 = \frac{32}{2} = 16$$
$$u_4 = \frac{16}{2} = 8$$

$u_5$ would be 4, but we stop at $u_4 = 8$.
Actually $u_0$ isn't defined.
First five: 64,32,16,8,4
$$\boxed{64,\ 32,\ 16,\ 8,\ 4}$$

**(c)** $u_0 = 0$
$$u_1 = 0 + 2(0) + 1 = 1$$
$$u_2 = 1 + 2(1) + 1 = 4$$
$$u_3 = 4 + 2(2) + 1 = 9$$
$$u_4 = 9 + 2(3) + 1 = 16$$

These are perfect squares: $u_n = n^2$
$$\boxed{0,\ 1,\ 4,\ 9,\ 16}$$

## Problem 8 :

### Answer:

**(a)** $u_4 = 13,\ u_9 = 28$

$u_9 - u_4 = 5d \Rightarrow 28 - 13 = 15 \Rightarrow 5d = 15 \Rightarrow d = 3$
$$u_1 = u_4 - 3d = 13 - 9 = 4$$
$$S_{10} = \frac{10(2u_1 + 9d)}{2} = 5(8 + 27) = 5(35) = 175$$
$$\boxed{d = 3,\ u_1 = 4,\ S_{10} = 175}$$

**(b)** $u_2 = 7,\ u_6 = 27$

$u_6 - u_2 = 4d \Rightarrow 27 - 7 = 20 \Rightarrow 4d = 20 \Rightarrow d = 5$
$$u_1 = u_2 - d = 7 - 5 = 2$$
$$S_{10} = \frac{10(2(2) + 9(5))}{2} = 5(4 + 45) = 5(49)$$
$$= 245$$
$$\boxed{d = 5,\ u_1 = 2,\ S_{10} = 245}$$

**(c)** $u_3 = 11,\ u_8 = 31$

$u_8 - u_3 = 5d \Rightarrow 31 - 11 = 20 \Rightarrow 5d = 20 \Rightarrow d = 4$
$$u_1 = u_3 - 2d = 11 - 8 = 3$$
$$S_{10} = \frac{10(2(3) + 9(4))}{2} = 5(6 + 36) = 5(42)$$
$$= 210$$
$$\boxed{d = 4,\ u_1 = 3,\ S_{10} = 210}$$

## Problem 9 :

### Answer:

**(a)** $u_2 = 6,\ u_5 = 48$

$u_5 = u_2 q^3 \Rightarrow 48 = 6q^3 \Rightarrow q^3 = 8 \Rightarrow q = 2$
$$u_1 = \frac{u_2}{q} = \frac{6}{2} = 3$$
$$S_5 = \frac{u_1(q^5 - 1)}{q - 1} = \frac{3(32 - 1)}{1} = 3(31) = 93$$
$$\boxed{q = 2,\ u_1 = 3,\ S_5 = 93}$$

**(b)** $u_1 = 81,\ u_4 = 3$

$u_4 = u_1 q^3 \Rightarrow 3 = 81q^3 \Rightarrow q^3 = \frac{1}{27} \Rightarrow q = \frac{1}{3}$
$$S_5 = \frac{81\left(1 - \left(\frac{1}{3}\right)^5\right)}{1 - \frac{1}{3}} = \frac{81\left(1 - \frac{1}{243}\right)}{\frac{2}{3}} = \frac{81 \cdot \frac{242}{243}}{\frac{2}{3}}$$
$$= \frac{81 \cdot 242}{243} \cdot \frac{3}{2} = \frac{242}{3} \cdot \frac{3}{2} = 121$$
$$\boxed{q = \frac{1}{3},\ u_1 = 81,\ S_5 = 121}$$

**(c)** $u_3 = 8,\ u_6 = 64$

$u_6 = u_3 q^3 \Rightarrow 64 = 8q^3 \Rightarrow q^3 = 8 \Rightarrow q = 2$
$$u_1 = \frac{u_3}{q^2} = \frac{8}{4} = 2$$
$$S_5 = \frac{2(2^5 - 1)}{2 - 1} = 2(32 - 1) = 62$$
$$\boxed{q = 2,\ u_1 = 2,\ S_5 = 62}$$

## Problem 10 :

### Answer :

**(a)** $u_n = 3n - 5$

$u_{n+1} - u_n = 3(n + 1) - 5 - (3n - 5) = 3 \rightarrow$ positive constant → **Increasing**
$$\boxed{\text{Increasing}}$$

**(b)** $u_n = (-1)^n$

Values: $-1, 1, -1, 1, \ldots$ (oscillates) → **Neither**
$$\boxed{\text{Neither}}$$

**(c)** $u_n = 2(0.5)^n$

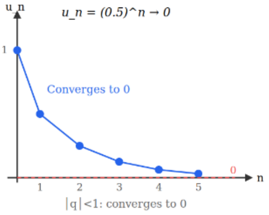
$u_{n+1}/u_n = 0.5 < 1$ and $u_1 > 0 \rightarrow$ **Decreasing**
$$\boxed{\text{Decreasing}}$$

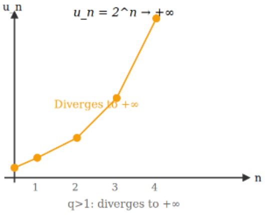
**(d)** Arithmetic: $u_1 = 10,\ d = -2$

$d < 0 \rightarrow$ **Decreasing**

$$\boxed{\text{Decreasing}}$$

## Problem 11 :





### Answer:

**(a)** $\lim_{n\to\infty}(0.4)^n = 0$ (since $| 0.4 |< 1$)

$$\boxed{0}$$

**(b)** $\lim_{n\to\infty} 3^n = +\infty$ (since $3 > 1$)

$$\boxed{+\infty}$$

**(c)** $\lim_{n\to\infty}(-0.9)^n = 0$ (since $| -0.9 |< 1$)

$$\boxed{0}$$

**(d)** $u_n = (-2)^n$ oscillates between positive and negative with increasing magnitude. No limit.

$$\boxed{\text{No limit}}$$

## Problem 12 :

### Answer: :

**(a)** Divide by $n$: $\frac{2+\frac{1}{n}}{1+\frac{3}{n}} \to \frac{2+0}{1+0} = 2$

$$\boxed{2}$$

**(b)** Divide by $n^2$: $\frac{3-\frac{1}{n}}{2+\frac{5}{n^2}} \to \frac{3-0}{2+0} = \frac{3}{2}$

$$\boxed{\frac{3}{2}}$$

**(c)** Divide by $n^2$: $\frac{\frac{1}{n}+\frac{1}{n^2}}{1+\frac{1}{n^2}} \to \frac{0+0}{1+0} = 0$

$$\boxed{0}$$

## Problem 13 :

### Answer: :

**(a)** $5n - 3 > 100 \Rightarrow 5n > 103 \Rightarrow n > 20.6 \Rightarrow n = 21$

$$\boxed{n = 21}$$

**(b)** $2^n > 1000 \Rightarrow 2^{10} = 1024 > 1000 \Rightarrow n = 10$

$$\boxed{n = 10}$$

**(c)** $1000(0.9)^n < 100 \Rightarrow (0.9)^n < 0.1$

Using logarithms (or trial): $(0.9)^{21} \approx 0.109$, $(0.9)^{22} \approx 0.098 \Rightarrow n = 22$

$$\boxed{n = 22}$$

## Problem 14 :

**Answer:** **(a)** Arithmetic with $u_0 = 7,\ d = 4$:

$$u_n = u_0 + nd = 7 + 4n$$

$$\boxed{u_n = 7 + 4n}$$

**(b)** Geometric with $u_1 = 5,\ q = 3$:

$$u_n = u_1 q^{n-1} = 5 \cdot 3^{n-1}$$

$$\boxed{u_n = 5 \cdot 3^{n-1}}$$

**(c)** Geometric with $u_0 = 100,\ q = 0.8$:

$$u_n = u_0 q^n = 100(0.8)^n$$

$$\boxed{u_n = 100(0.8)^n}$$

## Problem 15 :

### Answer:

**(a)** $\sum_{k=1}^{10} k = \frac{10(11)}{2} = 55$

$$\boxed{55}$$

**(b)** $\sum_{k=0}^{4} 2^k = 1 + 2 + 4 + 8 + 16 = 31$

$$\boxed{31}$$

**(c)** $\sum_{k=1}^{5}(2k - 1) = 1 + 3 + 5 + 7 + 9 = 25$

$$\boxed{25}$$

**(d)** $\sum_{k=1}^{n} 1 = n$

$$\boxed{n}$$

# Problems 16–38 — Medium

## Problem 16 :

### Answer:

**(a)** $u_0 = 20000$

After each year, the balance is multiplied by $1.04$:

$$u_{n+1} = 1.04u_n$$

$$\boxed{u_0 = 20000,\ u_{n+1} = 1.04u_n}$$

**(b)** $u_n = 20000(1.04)^n$

$$\boxed{u_n = 20000(1.04)^n}$$

**(c)** $20000(1.04)^n > 30000 \Rightarrow (1.04)^n > 1.5$

$n\ln(1.04) > \ln(1.5) \Rightarrow n > \frac{\ln(1.5)}{\ln(1.04)} \approx 10.34$

So $n = 11$ years

$$\boxed{\text{11 years}}$$

## Problem 17 :

### Answer:

**(a)** $u_{n+1} = 1.008u_n - 2000$

$$\boxed{u_{n+1} = 1.008u_n - 2000}$$

**(b)** $u_0 = 100000$

$$\begin{aligned} u_1 &= 1.008(100000) - 2000 = 100800 - 2000 \\ &= 98800 \\ u_2 &= 1.008(98800) - 2000 = 99590.4 - 2000 \\ &= 97590.4 \\ u_3 &= 1.008(97590.4) - 2000 \\ &= 98371.1232 - 2000 \\ &= 96371.1232 \end{aligned}$$

$$\boxed{\begin{array}{c} u_0 = 100000,\ u_1 = 98800,\ u_2 = 97590.4, \\ u_3 = 96371.12 \end{array}}$$

**(c)** The balance is decreasing each month (by about 1200 DA).
Check: 0.8% of 100000 = 800 < 2000 payment, so net decrease.

$$\boxed{\text{Yes, the loan is being paid off.}}$$

## Problem 18 :

**Answer:**

Arithmetic sequence: $u_1 = 500,\ d = 80.$

**(a)** $u_{12} = 500 + 11(80) = 500 + 880 = 1380$

$$\boxed{\text{1380 items}}$$

**(b)** $u_n = 500 + (n-1)80 > 1500 \Rightarrow (n-1)80 > 1000 \Rightarrow n - 1 > 12.5 \Rightarrow n = 14$

$$\boxed{\text{Month 14}}$$

**(c)** $u_{24} = 500 + 23(80) = 500 + 1840 = 2340$

$$S_{24} = \frac{24(500 + 2340)}{2} = 12(2840) = 34080$$

$$\boxed{\text{34080 items}}$$

## Problem 19 :

**Answer:**

**(a)** $u_n = 250000(1.025)^n$

$$\boxed{u_n = 250000(1.025)^n}$$

**(b)** $u_{10} = 250000(1.025)^{10} \approx 250000(1.28008) \approx 320020$

$$\boxed{\text{320020 (approx.)}}$$

**(c)** $250000(1.025)^n > 400000 \Rightarrow (1.025)^n > 1.6$
$n\ln(1.025) > \ln(1.6) \Rightarrow n > \frac{\ln(1.6)}{\ln(1.025)} \approx 18.95$
So $n = 19$ years after 2020 → **2039**

$$\boxed{2039}$$

## Problem 20 :

**Answer:**

**(a)** $u_1 = S_1 = 3(1)^2 + 2(1) = 3 + 2 = 5$

$$\boxed{u_1 = 5}$$

**(b)** For $n \geq 2$:

$$\begin{aligned} S_n &= 3n^2 + 2n \\ S_{n-1} &= 3(n-1)^2 + 2(n-1) \\ &= 3(n^2 - 2n + 1) + 2n - 2 \\ &= 3n^2 - 6n + 3 + 2n - 2 \\ &= 3n^2 - 4n + 1 \\ u_n = S_n - S_{n-1} &= (3n^2 + 2n) - (3n^2 - 4n + 1) \\ &= 6n - 1 \end{aligned}$$

$$\boxed{u_n = 6n - 1 \text{ for } n \geq 2}$$

**(c)** $d = u_{n+1} - u_n = 6(n+1) - 1 - (6n - 1) = 6n + 6 - 1 - 6n + 1 = 6$

$$\boxed{d = 6}$$

## Problem 21 :

**Answer:** For $u_n$:
$u_{n+1} - u_n = 5(n+1) - 3 - (5n - 3) = 5n + 5 - 3 - 5n + 3 = 5$ (constant).
So $(u_n)$ is arithmetic with $d = 5$.

$$\boxed{u_n \text{ is arithmetic}}$$

For $v_n$:

$$\begin{aligned} v_{n+1} - v_n &= (n+1)^2 + (n+1) - (n^2 + n) \\ &= n^2 + 2n + 1 + n + 1 - n^2 - n \\ &= 2n + 2 \end{aligned}$$

This depends on $n$, so it is not constant.

$$\boxed{v_n \text{ is not arithmetic}}$$

## Problem 22 :

**Answer:** For $u_n$:
$\frac{u_{n+1}}{u_n} = \frac{7\left(\frac{3}{2}\right)^{n+1}}{7\left(\frac{3}{2}\right)^n} = \frac{3}{2}$ (constant).
So $(u_n)$ is geometric with $q = \frac{3}{2}$.

$$\boxed{u_n \text{ is geometric}}$$

For $w_n$:

$$\begin{aligned} w_0 &= 4 \\ w_1 &= 2(4) - 1 = 7 \\ w_2 &= 2(7) - 1 = 13 \\ w_3 &= 2(13) - 1 = 25 \end{aligned}$$

Check ratios: $7/4 = 1.75$, $13/7 \approx 1.857$, $25/13 \approx 1.923$ → not constant.

$$\boxed{w_n \text{ is not geometric}}$$

.

## Problem 23 :

**Answer:**

Let $u_1 = 2$, $u_2 = 2 + d$, $u_3 = 2 + 2d$.
For the geometric sequence: $u_1, u_2, u_4$ with common ratio $q$.
$u_2 = u_1 q \Rightarrow 2 + d = 2q \Rightarrow q = \frac{2+d}{2}$
$u_4 = u_2 q = (2 + d)q$
But $u_4 = u_1 + 3d = 2 + 3d$ (since arithmetic).
So $(2 + d)\left(\frac{2+d}{2}\right) = 2 + 3d$
$\frac{(2+d)^2}{2} = 2 + 3d \Rightarrow (2 + d)^2 = 4 + 6d$
$4 + 4d + d^2 = 4 + 6d \Rightarrow d^2 - 2d = 0 \Rightarrow d(d - 2) = 0$
$d = 0$ (trivial) or $d = 2$.
With $d = 2$: $u_1 = 2,\ u_2 = 4,\ u_3 = 6$ (arithmetic), and $u_1 = 2,\ u_2 = 4,\ u_4 = 8$ (geometric). ✓

$$\boxed{d = 2}$$

## Problem 24 :

**Answer:**

**(a)** $S_3 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) = 1 - \frac{1}{4} = \frac{3}{4}$

$$\boxed{S_3 = \frac{3}{4}}$$

**(b)** $S_n = 1 - \frac{1}{n+1} = \frac{n}{n+1}$

$$\boxed{S_n = \frac{n}{n+1}}$$

**(c)** $\lim_{n\to\infty} \frac{n}{n+1} = 1$

$$\boxed{1}$$

## Problem 25 :

**Answer:**

**(a)** $v_{n+1} = u_{n+1} + 4 = (2u_n + 4) + 4 = 2u_n + 8 = 2(u_n + 4) = 2v_n$

$\frac{v_{n+1}}{v_n} = 2 \rightarrow$ geometric with $q = 2$.

$$\boxed{v_n \text{ is geometric with } q = 2}$$

**(b)** $v_0 = u_0 + 4 = 3 + 4 = 7$

$$v_n = 7 \cdot 2^n$$
$$u_n = v_n - 4 = 7 \cdot 2^n - 4$$
$$\boxed{u_n = 7 \cdot 2^n - 4}$$

**(c)** $u_0 = 7(1) - 4 = 3$ ✓

$u_1 = 7(2) - 4 = 14 - 4 = 10$ ✓

$u_2 = 7(4) - 4 = 28 - 4 = 24$ ✓

$$\boxed{\text{Verified}}$$

## Problem 26 :

### Answer:

**Base case:** $n = 1$

LHS: 1, RHS: $\frac{1(2)}{2} = 1$ ✓

**Inductive step:** Assume true for $n = k$:

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}$$

Add $k + 1$ to both sides:

$$1 + 2 + \cdots + k + (k+1) = \frac{k(k+1)}{2} + (k+1)$$
$$= (k+1)\left(\frac{k}{2} + 1\right)$$
$$= (k+1)\left(\frac{k+2}{2}\right)$$
$$= \frac{(k+1)(k+2)}{2}$$

This is the formula for $n = k + 1$. ✓

$$\boxed{\text{Proved by induction}}$$

## Problem 27 :

### Answer:

**Base case:** $n = 0$

LHS: 1, RHS: $\frac{1-q^1}{1-q} = 1$ ✓

**Inductive step:** Assume true for $n = k$:

$$1 + q + \cdots + q^k = \frac{1 - q^{k+1}}{1 - q}$$

Add $q^{k+1}$ to both sides:

$$1 + q + \cdots + q^k + q^{k+1} = \frac{1 - q^{k+1}}{1 - q} + q^{k+1}$$
$$= \frac{1 - q^{k+1} + q^{k+1}(1 - q)}{1 - q}$$
$$= \frac{1 - q^{k+1} + q^{k+1} - q^{k+2}}{1 - q}$$
$$= \frac{1 - q^{k+2}}{1 - q}$$

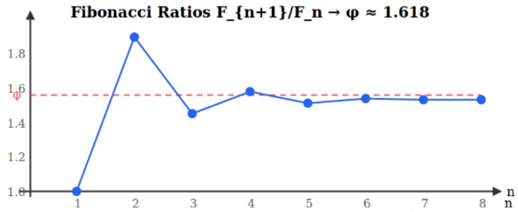
This is the formula for $n = k + 1$. ✓

$$\boxed{\text{Proved by induction}}$$

**(c)** $\sum_{k=0}^{5} \left(\frac{1}{2}\right)^k = \frac{1-(1/2)^6}{1-1/2} = \frac{1-1/64}{1/2} = \frac{63/64}{1/2} = \frac{63}{32}$

$$\boxed{\frac{63}{32}}$$

## Problem 28 :



The ratio oscillates and converges to the golden ratio φ = (1+√5)/2 ≈ 1.618

### Answer:

**(a)** $F_1 = 1,\ F_2 = 1$

$$F_3 = 1 + 1 = 2$$
$$F_4 = 2 + 1 = 3$$
$$F_5 = 3 + 2 = 5$$
$$F_6 = 5 + 3 = 8$$
$$F_7 = 8 + 5 = 13$$
$$F_8 = 13 + 8 = 21$$
$$F_9 = 21 + 13 = 34$$
$$F_{10} = 34 + 21 = 55$$
$$\boxed{1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55}$$

**(b)** Ratios:

$F_2/F_1 = 1,\ F_3/F_2 = 2,\ F_4/F_3 = 1.5,\ F_5/F_4 \approx 1.667,\ F_6/F_5 = 1.6,\ F_7/F_6 \approx 1.625,\ F_8/F_7 \approx 1.615,\ F_9/F_8 \approx 1.619$

$$\boxed{1,\ 2,\ 1.5,\ 1.667,\ 1.6,\ 1.625,\ 1.615,\ 1.619}$$

**(c)** $n = 1$: $F_1 = 1,\ F_3 - 1 = 2 - 1 = 1$ ✓

$n = 2$: $F_1 + F_2 = 2,\ F_4 - 1 = 3 - 1 = 2$ ✓

$n = 3$: $1 + 1 + 2 = 4,\ F_5 - 1 = 5 - 1 = 4$ ✓

$n = 4$: $1 + 1 + 2 + 3 = 7,\ F_6 - 1 = 8 - 1 = 7$ ✓

$$\boxed{\text{Verified for } n = 1{,}2{,}3{,}4}$$

## Problem 29 :

### Answer:

**(a)** Exponential grows faster than polynomial $\rightarrow \lim_{n\to\infty} \frac{3^n}{n^2} = +\infty$

$$\boxed{+\infty}$$

**(b)** Exponential ($2^n$) eventually dominates any polynomial $\rightarrow \lim_{n\to\infty} \frac{n^5}{2^n} = 0$

$$\boxed{0}$$

**(c)** Divide by $n^2$: $\frac{(1+1/n)^2}{1-1/n+1/n^2} \rightarrow \frac{1}{1} = 1$

$$\boxed{1}$$

**(d)** Geometric decay ($0.95^n$) beats linear growth $\rightarrow \lim_{n\to\infty} n(0.95)^n = 0$

$$\boxed{0}$$

## Problem 30 :

### Answer:

**(a)** $u_1 = S_1 = 2(1)^2 - 1 = 2 - 1 = 1$

$u_2 = S_2 - S_1 = [2(4) - 2] - 1 = 8 - 2 - 1 = 5$

$u_3 = S_3 - S_2 = [2(9) - 3] - 6 = 18 - 3 - 6 = 9$

$$\boxed{u_1 = 1,\ u_2 = 5,\ u_3 = 9}$$

**(b)** For $n \geq 2$:

$$u_n = S_n - S_{n-1} = (2n^2 - n) - [2(n-1)^2 - (n-1)]$$

$$= 2n^2 - n - [2(n^2 - 2n + 1) - n + 1]$$

$$= 2n^2 - n - [2n^2 - 4n + 2 - n + 1]$$

$$= 2n^2 - n - [2n^2 - 5n + 3]$$

$$= 2n^2 - n - 2n^2 + 5n - 3$$

$$= 4n - 3$$

So $u_n = 4n - 3$ for $n \geq 2$. Check $n = 1$: $4(1) - 3 = 1$ ✓

$$\boxed{u_n = 4n - 3}$$

**(c)** $d = u_{n+1} - u_n = 4(n+1) - 3 - (4n - 3) = 4n + 4 - 3 - 4n + 3 = 4$

$$\boxed{d = 4}$$

## Problem 31 :

### Answer:

**(a)** $u_n = 800000(0.85)^n$

$$\boxed{u_n = 800000(0.85)^n}$$

**(b)** $u_5 = 800000(0.85)^5 \approx 800000(0.4437) \approx 354960$ DA

$$\boxed{354960 \text{ DA}}$$

**(c)** $800000(0.85)^n < 200000 \Rightarrow (0.85)^n < 0.25$

$n\ln(0.85) < \ln(0.25)$ (note $\ln(0.85)$ is negative, so reverse inequality)

$$n > \frac{\ln(0.25)}{\ln(0.85)} \approx 8.53$$

So $n = 9$ years

$$\boxed{9 \text{ years}}$$

## Problem 32 :

### Answer:

Verification for $n = 4$:

$$1^2 + 2^2 + 3^2 + 4^2 = 1 + 4 + 9 + 16 = 30$$

$\frac{4(5)(9)}{6} = \frac{180}{6} = 30$ ✓

**(a)** $\sum_{k=1}^{10} k^2 = \frac{10(11)(21)}{6} = \frac{2310}{6} = 385$

$$\boxed{385}$$

**(b)** $\sum_{k=5}^{10} k^2 = \sum_{k=1}^{10} k^2 - \sum_{k=1}^{4} k^2 = 385 - 30 = 355$

$$\boxed{355}$$

**(c)** $\sum_{k=1}^{n}(k^2 + k) = \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k = \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{3n(n+1)}{6}$$

$$= \frac{n(n+1)(2n+1+3)}{6}$$

$$= \frac{n(n+1)(2n+4)}{6}$$

$$= \frac{n(n+1)(n+2)}{3}$$

$$\boxed{\frac{n(n+1)(n+2)}{3}}$$

## Problem 33 :

### Answer:

**(a)** $u_n = (-1)^n$ alternates between 1 and $-1$, so it has no limit.

$$\boxed{\text{No limit}}$$

**(b)** $|\ q\ | = \frac{1}{2} < 1$, so $\lim_{n\to\infty}\left(-\frac{1}{2}\right)^n = 0$

$$\boxed{0}$$

**(c)** $|\ u_n\ | = (0.9)^n \to 0$. Since $-(0.9)^n \leq u_n \leq (0.9)^n$, by the squeeze theorem, $u_n \to 0$.

$$\boxed{0}$$

## Problem 34 :

### Answer:

**(a)** $L = \frac{L}{3} + 4 \Rightarrow 3L = L + 12 \Rightarrow 2L = 12 \Rightarrow L = 6$

$$v_{n+1} = u_{n+1} - 6 = \left(\frac{u_n}{3} + 4\right) - 6 = \frac{u_n}{3} - 2 = \frac{u_n - 6}{3} = \frac{v_n}{3}$$

So $v_{n+1} = \frac{1}{3} v_n \to$ geometric with $q = \frac{1}{3}$.

$$\boxed{v_n \text{ is geometric with } q = \frac{1}{3}}$$

**(b)** $v_0 = u_0 - 6 = 2 - 6 = -4$

$$v_n = -4\left(\frac{1}{3}\right)^n$$

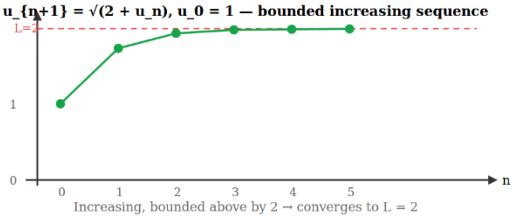
$$u_n = v_n + 6 = 6 - 4\left(\frac{1}{3}\right)^n$$

$$\boxed{u_n = 6 - 4\left(\frac{1}{3}\right)^n}$$

**(c)** $\lim_{n\to\infty} u_n = 6 - 4(0) = 6$

$$\boxed{6}$$

## Problem 35 :



### Answer:

**(a)** $u_0 = 1$

$$u_1 = \sqrt{2 + 1} = \sqrt{3} \approx 1.732$$

$$u_2 = \sqrt{2 + 1.732} = \sqrt{3.732} \approx 1.932$$

$$u_3 = \sqrt{2 + 1.932} = \sqrt{3.932} \approx 1.983$$

$$\boxed{1,\ 1.732,\ 1.932,\ 1.983}$$

**(b)** If $u_n < 2$, then $u_{n+1} = \sqrt{2 + u_n} < \sqrt{4} = 2$.

Also, $u_{n+1} > u_n$ because $\sqrt{2 + u_n} > u_n$ for $0 < u_n < 2$.

So increasing and bounded above by 2.

**(c)** $L = \sqrt{2 + L} \Rightarrow L^2 = 2 + L \Rightarrow L^2 - L - 2 = 0 \Rightarrow (L-2)(L+1) = 0 \Rightarrow L = 2$ (since positive).

$$\boxed{L = 2}$$

## Problem 36 :

**Answer:**

**(a)** $u_1 = S_1 = 3^1 - 1 = 3 - 1 = 2$

$$\boxed{u_1 = 2}$$

**(b)** $S_n = 3^n - 1$

$S_{n-1} = 3^{n-1} - 1$

$u_n = S_n - S_{n-1} = (3^n - 1) - (3^{n-1} - 1)$

$= 3^n - 3^{n-1} = 3^{n-1}(3 - 1)$

$= 2 \cdot 3^{n-1}$

For $n = 1$, $2 \cdot 3^0 = 2$ ✓

$$\boxed{u_n = 2 \cdot 3^{n-1}}$$

**(c)** $q = \frac{u_{n+1}}{u_n} = \frac{2 \cdot 3^n}{2 \cdot 3^{n-1}} = 3$

$$\boxed{q = 3}$$

## Problem 37 :

**Answer:**

**(a)** $u_5 = u_1 q^4 \Rightarrow 80 = 5q^4 \Rightarrow q^4 = 16 \Rightarrow q =$ 2 (positive ratio)

$$\boxed{q = 2}$$

**(b)** $v_n = \log_2(u_n)$

$v_{n+1} = \log_2(u_{n+1}) = \log_2(qu_n)$

$= \log_2(q) + \log_2(u_n)$

$= v_n + \log_2(2) = v_n + 1$

So $(v_n)$ is arithmetic with $d = 1$.

$$\boxed{d = 1}$$

**(c)** $v_1 = \log_2(u_1) = \log_2(5)$

$v_n = v_1 + (n - 1)d = \log_2(5) + (n - 1)$

$u_n = 2^{v_n} = 2^{\log_2(5)+n-1} = 5 \cdot 2^{n-1}$

$$\boxed{u_n = 5 \cdot 2^{n-1}}$$

## Problem 38 :

**Answer:**

**(a)** Divide by $n^2$: $\frac{4+\frac{3}{n}-\frac{1}{n^2}}{2+\frac{1}{n^2}} \to \frac{4}{2} = 2$

$$\boxed{2}$$

**(b)** $\frac{2^n+3^n}{4^n} = \left(\frac{2}{4}\right)^n + \left(\frac{3}{4}\right)^n = \left(\frac{1}{2}\right)^n + \left(\frac{3}{4}\right)^n \to 0 + 0 = 0$

$$\boxed{0}$$

**(c)** $n\left(\frac{2}{3}\right)^n \to 0$ (exponential decay beats linear growth)

$$\boxed{0}$$

**(d)** $\left(1+\frac{1}{n}\right)^n \to e \approx 2.71828$

$$\boxed{e}$$

# Problems 39–50 — Difficult & Real-World

## Problem 39 :

**Answer:**

**(a)** $u_5 = u_2 q^3 \Rightarrow 96 = 12q^3 \Rightarrow q^3 = 8 \Rightarrow q = 2$

$$u_1 = \frac{u_2}{q} = \frac{12}{2} = 6$$

$$\boxed{q = 2,\ u_1 = 6}$$

**(b)** $S_7 = \frac{6(2^7-1)}{2-1} = 6(128 - 1) = 6(127) = 762$

$$\boxed{762}$$

**(c)** $u_n = 6 \cdot 2^{n-1} > 10000 \Rightarrow 2^{n-1} > \frac{10000}{6} \approx$ 1666.67

$2^{10} = 1024$, $2^{11} = 2048 \Rightarrow n - 1 = 11 \Rightarrow n = 12$

$$\boxed{n = 12}$$

**(d)** $v_n = \log_2(u_n)$

$v_{n+1} - v_n = \log_2(u_{n+1}) - \log_2(u_n) = \log_2\left(\frac{u_{n+1}}{u_n}\right)$

$= \log_2(q) = \log_2(2) = 1$

So $(v_n)$ is arithmetic with $d = 1$.

$$\boxed{d = 1}$$

## Problem 40 :

**Answer:**

**(a)** $u_0 = 0$

$u_1 = 0 + 2(0) + 1 = 1$

$u_2 = 1 + 2(1) + 1 = 4$

$u_3 = 4 + 2(2) + 1 = 9$

$u_4 = 9 + 2(3) + 1 = 16$

$u_5 = 16 + 2(4) + 1 = 25$

Conjecture: $u_n = n^2$

$$\boxed{0,\ 1,\ 4,\ 9,\ 16,\ 25}$$

**(b)** Base case: $u_0 = 0 = 0^2$ ✓

Inductive step: Assume $u_k = k^2$.

Then $u_{k+1} = u_k + 2k + 1 = k^2 + 2k + 1 = (k + 1)^2$ ✓

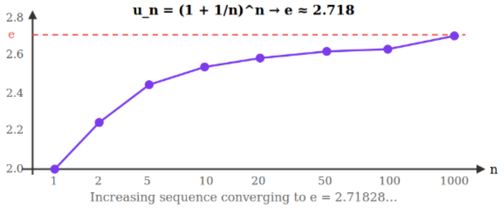
So $u_n = n^2$ for all $n$.

$$\boxed{\text{Proved by induction}}$$

**(c)** $\sum_{n=0}^{9} u_n = \sum_{n=0}^{9} n^2 = \frac{9(10)(19)}{6} = \frac{1710}{6} = 285$

$$\boxed{285}$$

## Problem 41 :



**Answer:**

**(a)** $u_1 = (1 + 1)^1 = 2$

$$u_{10} = (1.1)^{10} \approx 2.5937$$
$$u_{100} = (1.01)^{100} \approx 2.7048$$
$$u_{1000} = (1.001)^{1000} \approx 2.7169$$
$$\boxed{2,\ 2.5937,\ 2.7048,\ 2.7169}$$

**(b)** The sequence is increasing and bounded above (by 3).

**(c)** As $n \to \infty$, $u_n \to e \approx 2.71828$.

$$\boxed{e}$$

## Problem 42 :

### Answer:

**(a)** $(\sqrt{a} - \sqrt{b})^2 = a - 2\sqrt{ab} + b \geq 0$

**(b)** $a + b \geq 2\sqrt{ab} \Rightarrow \frac{a+b}{2} \geq \sqrt{ab}$

$$\boxed{A \geq G}$$

**(c)** Equality occurs when $\sqrt{a} = \sqrt{b}$, i.e., $a = b$.

**(d)** For $a = 3$, $b = 12$:

$A = \frac{3+12}{2} = 7.5$, $G = \sqrt{36} = 6$, $A > G$ ✓

$$\boxed{\text{Equality iff } a = b}$$

## Problem 43 :

### Answer:

**(a)** $u_n = 500 \cdot 2^n$

$$\boxed{u_n = 500 \cdot 2^n}$$

**(b)** 24 hours = 8 periods: $u_8 = 500 \cdot 2^8 = 500 \cdot 256 = 128000$

$$\boxed{128000 \text{ cells}}$$

**(c)** $500 \cdot 2^n > 1\,000\,000 \Rightarrow 2^n > 2000$

$2^{10} = 1024$, $2^{11} = 2048 \Rightarrow n = 11$ periods = 33 hours

$$\boxed{33 \text{ hours}}$$

**(d)** Growth 200% means multiply by 2. Death 10% means multiply by 0.9.

Net growth factor: $2 \cdot 0.9 = 1.8$

$$u_{n+1} = 1.8u_n$$
$$\boxed{u_{n+1} = 1.8u_n}$$

## Problem 44 :

### Answer:

**(a)** $u_1 = 1$

$$u_2 = 1 + \frac{1}{1+1} = 1 + \frac{1}{2} = 1.5$$
$$u_3 = 1 + \frac{1}{1+1.5} = 1 + \frac{1}{2.5} = 1.4$$
$$u_4 = 1 + \frac{1}{1+1.4} = 1 + \frac{1}{2.4} \approx 1.4167$$
$$u_5 = 1 + \frac{1}{1+1.4167} = 1 + \frac{1}{2.4167} \approx 1.4138$$
$$\boxed{1,\ 1.5,\ 1.4,\ 1.4167,\ 1.4138}$$

**(b)** $L = 1 + \frac{1}{1+L} \Rightarrow L(1+L) = (1+L) + 1 \Rightarrow L + L^2 = L + 2 \Rightarrow L^2 = 2$

$$\boxed{L = \sqrt{2} \approx 1.4142}$$

**(c)** $L^2 = 2 \Rightarrow L = \sqrt{2}$ (since $L > 0$).

$$\boxed{\sqrt{2}}$$

## Problem 45 :

### Answer:

**(a)** $H_1 = 1$

$$H_2 = 1 + \frac{1}{2} = 1.5$$
$$H_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 2$$
$$H_8 > 2 + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = 2 + 0.5 = 2.5$$
$$\boxed{1,\ 1.5,\ > 2,\ > 2.5}$$

**(b)** Group terms:

For $k = 1$: $H_2 = 1 + \frac{1}{2} \geq 1 + \frac{1}{2} = 1.5$ ✓

Assume true for $k$: $H_{2^k} \geq 1 + \frac{k}{2}$.

Then $H_{2^{k+1}} = H_{2^k} + \left(\frac{1}{2^k+1} + \cdots + \frac{1}{2^{k+1}}\right) \geq H_{2^k} + 2^k \cdot \frac{1}{2^{k+1}} = H_{2^k} + \frac{1}{2} \geq 1 + \frac{k+1}{2}$. ✓

$$\boxed{\text{Proved by induction}}$$

**(c)** As $k \to \infty$, $1 + \frac{k}{2} \to \infty$.

So $H_n \to +\infty$.

$$\boxed{H_n \to +\infty \text{ (diverges)}}$$

## Problem 46 :

### Answer:

**(a)** $S_n = u_1 \frac{1-q^n}{1-q}$

Since $\mid q \mid < 1$, $q^n \to 0$.

So $S = \lim\limits_{n \to \infty} S_n = \frac{u_1}{1-q}$. ✓

$$\boxed{S = \frac{u_1}{1-q}}$$

**(b)** $u_1 = 1$, $q = \frac{1}{3}$

$$S = \frac{1}{1 - 1/3} = \frac{1}{2/3} = \frac{3}{2}$$
$$\boxed{\frac{3}{2}}$$

**(c)** $0.333 \ldots = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \cdots$

$$u_1 = \frac{3}{10},\ q = \frac{1}{10}$$
$$S = \frac{3/10}{1 - 1/10} = \frac{3/10}{9/10} = \frac{1}{3}$$
$$\boxed{\frac{1}{3}}$$

**(d)** $0.121212 \ldots = \frac{12}{100} + \frac{12}{10000} + \cdots$

$$u_1 = \frac{12}{100},\ q = \frac{1}{100}$$
$$S = \frac{12/100}{1 - 1/100} = \frac{12/100}{99/100} = \frac{12}{99} = \frac{4}{33}$$
$$\boxed{\frac{4}{33}}$$

## Problem 47 :

### Answer:

**(a)** $u_0 = 1$

$$u_1 = 3$$
$$u_2 = 4(3) - 3(1) = 12 - 3 = 9$$
$$u_3 = 4(9) - 3(3) = 36 - 9 = 27$$
$$u_4 = 4(27) - 3(9) = 108 - 27 = 81$$
$$u_5 = 4(81) - 3(27) = 324 - 81 = 243$$
$$\boxed{1,\ 3,\ 9,\ 27,\ 81,\ 243}$$

**(b)** $r^2 - 4r + 3 = 0 \Rightarrow (r-1)(r-3) = 0 \Rightarrow r = 1,\ 3$

$$\boxed{r = 1,\ 3}$$

**(c)** $u_n = A \cdot 1^n + B \cdot 3^n = A + B \cdot 3^n$
Using initial conditions:
$n = 0$: $A + B = 1$
$n = 1$: $A + 3B = 3$
Subtract: $2B = 2 \Rightarrow B = 1, A = 0$
So $u_n = 3^n$

$$\boxed{u_n = 3^n}$$

**(d)** $u_5 = 3^5 = 243$ ✓

$$\boxed{\text{Verified}}$$

## Problem 48 :

**Answer:**

**(a)** $L = 1.03L - 2000 \Rightarrow -0.03L = -2000 \Rightarrow L = \frac{2000}{0.03} = \frac{200000}{3} \approx 66666.67$

$$\boxed{L = \frac{200000}{3}}$$

**(b)** $v_{n+1} = u_{n+1} - L = (1.03u_n - 2000) - L = 1.03u_n - (2000 + L)$
But $L = 1.03L - 2000 \Rightarrow 2000 + L = 1.03L$
So $v_{n+1} = 1.03u_n - 1.03L = 1.03(u_n - L) = 1.03v_n$
Thus $v_n$ is geometric with $q = 1.03$.

$$\boxed{v_{n+1} = 1.03v_n}$$

**(c)** $v_0 = u_0 - L = 80000 - \frac{200000}{3} = \frac{240000-200000}{3} = \frac{40000}{3}$

$$v_n = \frac{40000}{3}(1.03)^n$$
$$u_n = L + v_n = \frac{200000}{3} + \frac{40000}{3}(1.03)^n$$
$$\boxed{u_n = \frac{200000 + 40000(1.03)^n}{3}}$$

**(d)** Since $1.03 > 1$, $u_n \to +\infty$.
The population grows in the long term (growth exceeds emigration).

$$\boxed{+\infty}$$

## Problem 49 :

**Answer:**

**(a)** $v_{n+1} = \ln(u_{n+1}) = \ln(qu_n) = \ln(q) + \ln(u_n) = v_n + \ln(q)$
So $(v_n)$ is arithmetic with common difference $d = \ln(q)$.

$$\boxed{d = \ln(q)}$$

**(b)** If $(u_n)$ is increasing, then $q > 1$.
$u_1 = e^2 \Rightarrow v_1 = \ln(e^2) = 2$
Let $q = e$ so $d = \ln(e) = 1$
Then $v_n = 2 + (n-1) = n + 1$

$$u_n = e^{v_n} = e^{n+1}$$
$$\boxed{u_n = e^{n+1}}$$

**(c)** $u_n > e^{20} \Rightarrow e^{n+1} > e^{20} \Rightarrow n + 1 > 20 \Rightarrow n > 19 \Rightarrow n = 20$

$$\boxed{n = 20}$$

**(d)** $P_n = u_1u_2 \cdots u_n = e^{v_1}e^{v_2} \cdots e^{v_n} = e^{v_1+v_2+\cdots+v_n}$
Sum of arithmetic series: $\sum_{k=1}^{n} v_k = \sum_{k=1}^{n}(k+1) = \frac{n(n+1)}{2} + n = \frac{n^2+n+2n}{2} = \frac{n(n+3)}{2}$
So $P_n = e^{n(n+3)/2}$

$$\boxed{P_n = e^{n(n+3)/2}}$$

## Problem 50 :

**Answer:**

**(a)** $u_0 = 2$

$$u_1 = \frac{2+6}{2+2} = \frac{8}{4} = 2$$
$$u_2 = \frac{2+6}{2+2} = 2$$
$$u_3 = 2$$

The sequence is constant 2!
But the fixed point equation: $L = \frac{L+6}{L+2} \Rightarrow L^2 + 2L = L + 6 \Rightarrow L^2 + L - 6 = 0 \Rightarrow (L+3)(L-2) = 0 \Rightarrow L = 2$ (positive).
So the limit is 2, not $\sqrt{6}$.

$$\boxed{u_n = 2 \text{ for all } n,\ \lim = 2}$$

**(b)** If $u_n = \sqrt{6}$:

$$u_{n+1} = \frac{\sqrt{6}+6}{\sqrt{6}+2}$$

Rationalise: not equal to $\sqrt{6}$ generally. Let's check:
$\sqrt{6}(\sqrt{6}+2) = 6 + 2\sqrt{6}$, but numerator is $\sqrt{6} + 6$.
They are equal only if $6 + 2\sqrt{6} = 6 + \sqrt{6} \Rightarrow \sqrt{6} = 0$, false.
So $\sqrt{6}$ is **not** a fixed point of this recurrence.
The fixed point is $L = 2$.

**(c)** Let's compute $w_n = \frac{u_n-2}{u_n+2}$ (using $L = 2$).

$$w_{n+1} = \frac{u_{n+1}-2}{u_{n+1}+2} = \frac{\frac{u_n+6}{u_n+2} - 2}{\frac{u_n+6}{u_n+2} + 2} = \frac{u_n + 6 - 2u_n - 4}{u_n + 6 + 2u_n + 4}$$
$$= \frac{2 - u_n}{3u_n + 10}$$

This is not a simple multiple of $w_n$.
The recurrence $u_{n+1} = \frac{u_n+6}{u_n+2}$ has fixed point $L = 2$.
Since $u_0 = 2$, the sequence is constant.
**(d)** Since $u_n = 2$ for all $n$, $\lim_{n\to\infty} u_n = 2$.

$$\boxed{2}$$

# SOLUTIONS TO PROBLEMS

## Problems 1–10 — Instant Recall

### Problem 1 :

**Answer:**

**(a)** $120^\circ = 120 \cdot \frac{\pi}{180} = \frac{2\pi}{3}$

**(b)** $270^\circ = 270 \cdot \frac{\pi}{180} = \frac{3\pi}{2}$

**(c)** $15^\circ = 15 \cdot \frac{\pi}{180} = \frac{\pi}{12}$

**(d)** $\frac{7\pi}{4} = \frac{7\pi}{4} \cdot \frac{180}{\pi} = 7 \cdot 45 = 315^\circ$

**(e)** $\frac{5\pi}{6} = \frac{5\pi}{6} \cdot \frac{180}{\pi} = 5 \cdot 30 = 150^\circ$

**(f)** $\frac{11\pi}{3} = \frac{11\pi}{3} \cdot \frac{180}{\pi} = 11 \cdot 60 = 660^\circ$

$$\boxed{\text{(a) } \frac{2\pi}{3} \text{(b) } \frac{3\pi}{2} \text{(c) } \frac{\pi}{12} \text{(d) } 315^\circ \text{(e) } 150^\circ \text{(f) } 660^\circ}$$

### Problem 2 :

**Answer:**

**(a)** $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

**(b)** $\cos\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$ (QII, reference $\pi/4$)

**(c)** $\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

**(d)** $\sin\left(\frac{5\pi}{6}\right) = \sin\left(\pi - \frac{\pi}{6}\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$

**(e)** $\cos\left(\frac{7\pi}{4}\right) = \cos\left(2\pi - \frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

**(f)** $\tan\left(\frac{4\pi}{3}\right) = \tan\left(\pi + \frac{\pi}{3}\right) = \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$

$$\boxed{\text{(a) } \frac{\sqrt{3}}{2} \text{(b) } -\frac{\sqrt{2}}{2} \text{(c) } \frac{\sqrt{3}}{3} \text{(d) } \frac{1}{2} \text{(e) } \frac{\sqrt{2}}{2} \text{(f) } \sqrt{3}}$$

### Problem 3 :

**Answer:**

**(a)** $\sin\theta = \frac{1}{2} \rightarrow$ reference angle $\frac{\pi}{6}$.
Sine positive in QI and QII: $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

**(b)** $\cos\theta = -1 \rightarrow \theta = \pi$

**(c)** $\tan\theta = 0 \rightarrow \sin\theta = 0 \rightarrow \theta = 0, \pi$

**(d)** $\sin\theta = -\frac{\sqrt{2}}{2} \rightarrow$ reference angle $\frac{\pi}{4}$.
Sine negative in QIII and QIV: $\theta = \frac{5\pi}{4}, \frac{7\pi}{4}$

$$\boxed{\text{(a) } \frac{\pi}{6}, \frac{5\pi}{6} \text{(b) } \pi \text{(c) } 0, \pi \text{(d) } \frac{5\pi}{4}, \frac{7\pi}{4}}$$

### Problem 4 :

**Answer:**

**(a)** $200^\circ$ is in QIII. Sine is negative in QIII.

⊟

**(b)** $130^\circ$ is in QII. Cosine is negative in QII.

⊟

**(c)** $310^\circ$ is in QIV. Tangent is negative in QIV.

⊟

**(d)** $\sin(-40^\circ) = -\sin(40^\circ)$. Sine is positive in QI, so the result is negative.

⊟

### Problem 5 :

**Answer:**

**(a)** $\cos^2\theta = 1 - \sin^2\theta = 1 - \frac{9}{25} = \frac{16}{25}$.
In QI, $\cos\theta = \frac{4}{5}$.
$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{3/5}{4/5} = \frac{3}{4}$.

**(b)** $\sin^2\theta = 1 - \cos^2\theta = 1 - \frac{25}{169} = \frac{144}{169}$.
In QIII, sine is negative: $\sin\theta = -\frac{12}{13}$.
$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-12/13}{-5/13} = \frac{12}{5}$.

**(c)** $\tan\theta = 2$. In QIII, both sine and cosine are negative.
$\sec^2\theta = 1 + \tan^2\theta = 1 + 4 = 5 \Rightarrow \cos^2\theta = \frac{1}{5}$.
In QIII, $\cos\theta = -\frac{1}{\sqrt{5}}$.
$\sin\theta = \tan\theta\cos\theta = 2\left(-\frac{1}{\sqrt{5}}\right) = -\frac{2}{\sqrt{5}}$.

$$\boxed{\begin{array}{c} \text{(a) } \cos = \frac{4}{5}, \tan = \frac{3}{4} \\ \text{(b) } \sin = -\frac{12}{13}, \tan = \frac{12}{5} \\ \text{(c) } \cos = -\frac{1}{\sqrt{5}}, \sin = -\frac{2}{\sqrt{5}} \end{array}}$$

### Problem 6 :

**Answer:**

**(a)** $150^\circ$ in QII. Reference: $180^\circ - 150^\circ = 30^\circ$.
$\sin(150^\circ) = \sin(30^\circ) = \frac{1}{2}$.

**(b)** $225^\circ$ in QIII. Reference: $225^\circ - 180^\circ = 45^\circ$.
$\cos(225^\circ) = -\cos(45^\circ) = -\frac{\sqrt{2}}{2}$.

**(c)** $\tan(-60^\circ) = -\tan(60^\circ) = -\sqrt{3}$.

**(d)** $\sin(390^\circ) = \sin(390^\circ - 360^\circ) = \sin(30^\circ) = \frac{1}{2}$.

$$\boxed{\text{(a) } \frac{1}{2} \text{(b) } -\frac{\sqrt{2}}{2} \text{(c) } -\sqrt{3} \text{(d) } \frac{1}{2}}$$

## Problem 7 :

**Answer:**

**(a)** $\sin x$: Period $2\pi$, amplitude 1, domain $\mathbb{R}$, range $[-1,1]$.

**(b)** $\cos x$: Period $2\pi$, amplitude 1, domain $\mathbb{R}$, range $[-1,1]$.

**(c)** $\tan x$: Period $\pi$, no amplitude (unbounded), domain $\mathbb{R} \setminus \{\frac{\pi}{2} + k\pi\}$, range $\mathbb{R}$.

$$\boxed{\begin{array}{c}\text{(a) } 2\pi,\ 1,\ \mathbb{R},\ [-1,1] \\ \text{(b) } 2\pi,\ 1,\ \mathbb{R},\ [-1,1] \\ \text{(c) } \pi, \text{ no amp, } \mathbb{R} \setminus \{\frac{\pi}{2} + k\pi\},\ \mathbb{R}\end{array}}$$

## Problem 8 :

**Answer:**

**(a)** By definition on the unit circle, $\cos\theta = x$ and $\sin\theta = y$, and $x^2 + y^2 = 1$.

$$\boxed{\sin^2\theta + \cos^2\theta = 1}$$

**(b)** Divide both sides of (a) by $\cos^2\theta$: $\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \Rightarrow \tan^2\theta + 1 = \sec^2\theta$.

**(c)** On the unit circle, the point at angle $-\theta$ has coordinates $(\cos\theta, -\sin\theta)$.
So $\sin(-\theta) = -\sin\theta$.

**(d)** $\cos(\pi - \theta) = \cos\pi\cos\theta + \sin\pi\sin\theta = (-1)\cos\theta + 0 = -\cos\theta$.

## Problem 9 :

**Answer:**

**(a)** $2\sin x - 1 = 0 \Rightarrow \sin x = \frac{1}{2}$

Solutions: $x = \frac{\pi}{6}, \frac{5\pi}{6}$

**(b)** $\cos x + 1 = 0 \Rightarrow \cos x = -1$
Solution: $x = \pi$

**(c)** $\tan x = \sqrt{3} \Rightarrow \tan x = \tan\left(\frac{\pi}{3}\right)$

Solutions: $x = \frac{\pi}{3}, \frac{4\pi}{3}$ (since tangent has period $\pi$)

**(d)** $2\cos^2 x - 1 = 0 \Rightarrow \cos^2 x = \frac{1}{2} \Rightarrow \cos x = \pm\frac{\sqrt{2}}{2}$

Solutions: $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

$$\boxed{\text{(a) } \frac{\pi}{6}, \frac{5\pi}{6} \text{(b) } \pi \text{(c) } \frac{\pi}{3}, \frac{4\pi}{3} \text{(d) } \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}}$$

## Problem 10 :

**Answer:**

**(a)** $\sin\left(-\frac{\pi}{3}\right) = -\sin\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

**(b)** $\cos\left(-\frac{2\pi}{3}\right) = \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$

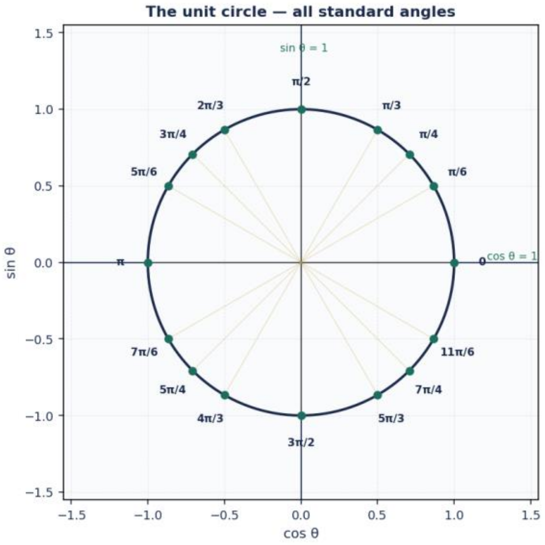
**(c)** $\sin\left(\frac{13\pi}{6}\right) = \sin\left(2\pi + \frac{\pi}{6}\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$

**(d)** $\cos\left(-\frac{7\pi}{4}\right) = \cos\left(\frac{7\pi}{4}\right) = \cos\left(2\pi - \frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$$\boxed{\text{(a) } -\frac{\sqrt{3}}{2} \text{(b) } -\frac{1}{2} \text{(c) } \frac{1}{2} \text{(d) } \frac{\sqrt{2}}{2}}$$

# Problems 11–35 — Medium

## Problem 11 :



**Answer:**

**(a)** At $\theta = \frac{2\pi}{3}$: coordinates $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

At $\theta = \frac{5\pi}{4}$: coordinates $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$.

**(b)** $\sin\theta = -\frac{1}{2}$ at $\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$.

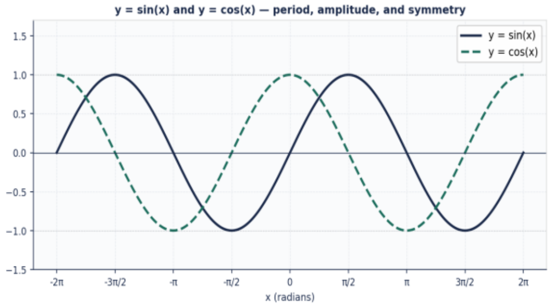
**(c)** $\cos\theta = \frac{\sqrt{2}}{2}$ at $\theta = \frac{\pi}{4}, \frac{7\pi}{4}$.

**(d)** $\sin\theta = \cos\theta$ when $\tan\theta = 1$: $\theta = \frac{\pi}{4}, \frac{5\pi}{4}$.

Coordinates: $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ and $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$.

$$\boxed{\begin{array}{c}\text{(a) } \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) \\ \text{(b) } \frac{7\pi}{6}, \frac{11\pi}{6} \\ \text{(c) } \frac{\pi}{4}, \frac{7\pi}{4} \\ \text{(d) } (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}), (-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})\end{array}}$$

## Problem 12 :



**Answer:**

**(a)** Zeros of $\sin x$ on $[-2\pi, 2\pi]$: $-2\pi, -\pi, 0, \pi, 2\pi$.
Maxima: at $x = -\frac{3\pi}{2}, \frac{\pi}{2}$ (value 1).
Minima: at $x = -\frac{\pi}{2}, \frac{3\pi}{2}$ (value -1).

**(b)** $\sin x = \cos x$ when $\tan x = 1$: $x = \frac{\pi}{4}, \frac{5\pi}{4}$ on $[0, 2\pi]$.

**(c)** $\sin x$ is odd; $\cos x$ is even.

**(d)** $\sin x > \cos x$ on $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$.

$$\boxed{\begin{array}{c}\text{(a) Zeros: } -2\pi, -\pi, 0, \pi, 2\pi; \\ \text{Max at } -\dfrac{3\pi}{2}, \dfrac{\pi}{2}; \text{ Min at } -\dfrac{\pi}{2}, \dfrac{3\pi}{2} \\ \text{(b) } \dfrac{\pi}{4}, \dfrac{5\pi}{4} \\ \text{(c) sin odd, cos even} \\ \text{(d) } (\dfrac{\pi}{4}, \dfrac{5\pi}{4})\end{array}}$$

## Problem 13 :

**Answer:**

**(a)** $1 - \cos^2\theta = \sin^2\theta$
$\frac{\sin^2\theta}{\sin\theta} = \sin\theta$ ✓

**(b)** Cross-multiply:
$\sin^2\theta = (1 - \cos\theta)(1 + \cos\theta) = 1 - \cos^2\theta = \sin^2\theta$ ✓

**(c)** $\tan\theta + \frac{1}{\tan\theta} = \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} = \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta} = \frac{1}{\sin\theta\cos\theta}$ ✓

## Problem 14 :

**Answer:**

**(a)** $\sin(45^\circ + 30^\circ) = \sin 45^\circ\cos 30^\circ + \cos 45^\circ\sin 30^\circ$

$$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6} + \sqrt{2}}{4}$$

**(b)** $\cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \cos\frac{\pi}{4}\cos\frac{\pi}{6} + \sin\frac{\pi}{4}\sin\frac{\pi}{6}$

$$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6} + \sqrt{2}}{4}$$

**(c)** $\tan(60^\circ + 45^\circ) = \frac{\tan 60^\circ + \tan 45^\circ}{1 - \tan 60^\circ \tan 45^\circ} = \frac{\sqrt{3}+1}{1-\sqrt{3}}$

Rationalise: $\frac{(\sqrt{3}+1)^2}{1-3} = \frac{4+2\sqrt{3}}{-2} = -2 - \sqrt{3}$

$$\boxed{\text{(a) } \frac{\sqrt{6} + \sqrt{2}}{4} \text{ (b) } \frac{\sqrt{6} + \sqrt{2}}{4} \text{ (c) } -2 - \sqrt{3}}$$

## Problem 15 :

**Answer:**

**(a)** $\cos\theta = \frac{4}{5}$ (from Problem 5).

$$\sin(2\theta) = 2\sin\theta\cos\theta = 2\left(\frac{3}{5}\right)\left(\frac{4}{5}\right) = \frac{24}{25}$$
$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = \frac{16}{25} - \frac{9}{25} = \frac{7}{25}$$
$$\tan(2\theta) = \frac{\sin(2\theta)}{\cos(2\theta)} = \frac{24}{7}$$

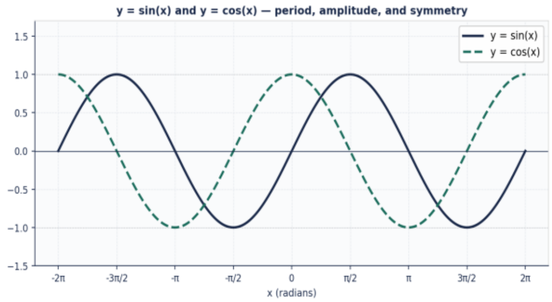
**(b)** $2\sin x\cos x = \sin(2x)$

**(c)** $\cos^2 x - \sin^2 x = \cos(2x)$

**(d)** $1 - 2\sin^2 x = \cos(2x)$

$$\boxed{\begin{array}{c}\text{(a) } \sin 2\theta = \dfrac{24}{25}, \cos 2\theta = \dfrac{7}{25}, \tan 2\theta = \dfrac{24}{7} \\ \text{(b) } \sin 2x \\ \text{(c) } \cos 2x \\ \text{(d) } \cos 2x\end{array}}$$

## Problem 16 :



**Answer:**

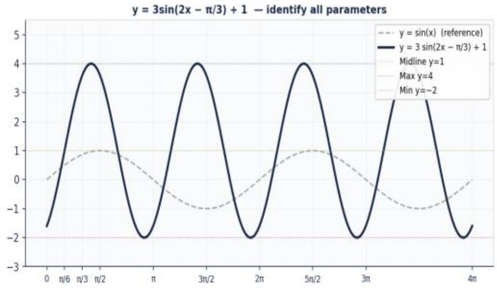
**(a)** $\sin x$ increases on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $\left[\frac{3\pi}{2}, \frac{5\pi}{2}\right]$.
On $[-2\pi, 2\pi]$: $\sin x$ increases on $\left[-\frac{3\pi}{2}, -\frac{\pi}{2}\right]$ and $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$.

**(b)** $\cos x$ decreases on $[0, \pi]$ and $[-2\pi, -\pi]$ (on $[-2\pi, 2\pi]$).

**(c)** When $\sin x$ increases, $\cos x$ decreases (they are $\pi/2$-phase shifted).

**(d)** $\sin x$ maximum = 1 at $x = \frac{\pi}{2}$; minimum = -1 at $x = \frac{3\pi}{2}$.

## Problem 17 :



**Answer:**

**(a)** Amplitude: $A = \frac{\text{max}-\text{min}}{2} = \frac{4-(-2)}{2} = 3$

**(b)** Period: $T = \pi$, so $\omega = \frac{2\pi}{T} = 2$

**(c)** Vertical shift: $D = \frac{\text{max}+\text{min}}{2} = \frac{4+(-2)}{2} = 1$

**(d)** First maximum at $x = \frac{5\pi}{12}$. For $y = A\sin(\omega x - \varphi) + D$:
$2\left(\frac{5\pi}{12}\right) - \varphi = \frac{\pi}{2} \Rightarrow \frac{5\pi}{6} - \varphi = \frac{\pi}{2} \Rightarrow \varphi = \frac{\pi}{3}$
Equation: $y = 3\sin\left(2x - \frac{\pi}{3}\right) + 1$

$$\boxed{A = 3,\ \omega = 2,\ D = 1,\ \varphi = \frac{\pi}{3},\ y = 3\sin(2x - \frac{\pi}{3}) + 1}$$

## Problem 18 :

**Answer:**

**(a)** Let $u = \sin x$. $2u^2 - u - 1 = 0 \Rightarrow (2u + 1)(u - 1) = 0$
$u = 1 \Rightarrow \sin x = 1 \Rightarrow x = \frac{\pi}{2}$
$u = -\frac{1}{2} \Rightarrow \sin x = -\frac{1}{2} \Rightarrow x = \frac{7\pi}{6}, \frac{11\pi}{6}$

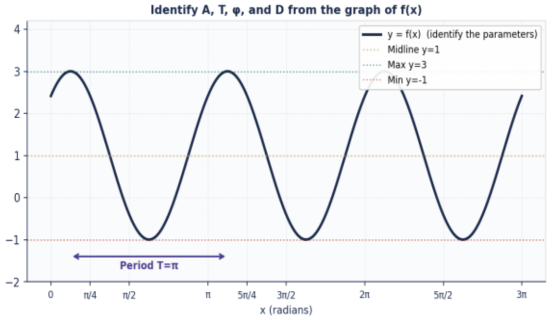
**(b)** Let $u = \cos x$. $2u^2 + 3u + 1 = 0 \Rightarrow (2u + 1)(u + 1) = 0$
$u = -1 \Rightarrow \cos x = -1 \Rightarrow x = \pi$
$u = -\frac{1}{2} \Rightarrow \cos x = -\frac{1}{2} \Rightarrow x = \frac{2\pi}{3}, \frac{4\pi}{3}$

**(c)** $\sin x = \cos x \Rightarrow \tan x = 1$ (provided $\cos x \neq 0$)

$$x = \frac{\pi}{4}, \frac{5\pi}{4}$$

$$\boxed{\text{(a) } \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6} \text{ (b) } \pi, \frac{2\pi}{3}, \frac{4\pi}{3} \text{ (c) } \frac{\pi}{4}, \frac{5\pi}{4}}$$

## Problem 19 :


Identify A, T, φ, and D from the graph of f(x)

**Answer:**

**(a)** Max = 3, Min = -1.

**(b)** Amplitude: $A = \frac{3-(-1)}{2} = 2$
Vertical shift: $D = \frac{3+(-1)}{2} = 1$

**(c)** Period: $T = \pi \Rightarrow \omega = \frac{2\pi}{\pi} = 2$

**(d)** First maximum at $x = \frac{\pi}{4}$. Using cosine form $y = A\cos(\omega x - \varphi) + D$:
$\omega \cdot \frac{\pi}{4} - \varphi = 0 \Rightarrow 2 \cdot \frac{\pi}{4} - \varphi = 0 \Rightarrow \varphi = \frac{\pi}{2}$
Equation: $y = 2\cos\left(2x - \frac{\pi}{2}\right) + 1$

Equivalently: $y = 2\sin(2x) + 1$ (since $\cos(2x - \pi/2) = \sin 2x$)

$$\boxed{y = 2\cos\left(2x - \frac{\pi}{2}\right) + 1 = 2\sin(2x) + 1}$$

## Problem 20 :

**Answer:**

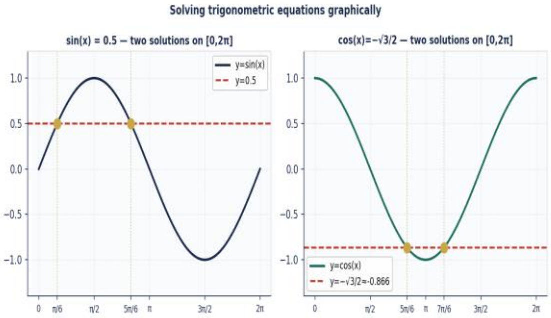
**(a)** $\sin x = \frac{\sqrt{3}}{2} \Rightarrow x = \frac{\pi}{3} + 2k\pi$ or $x = \frac{2\pi}{3} + 2k\pi$, $k \in \mathbb{Z}$.

**(b)** $\cos(2x) = \frac{1}{2} \Rightarrow 2x = \frac{\pi}{3} + 2k\pi$ or $2x = \frac{5\pi}{3} + 2k\pi$
$\Rightarrow x = \frac{\pi}{6} + k\pi$ or $x = \frac{5\pi}{6} + k\pi$, $k \in \mathbb{Z}$.

**(c)** $\tan x = -1 \Rightarrow x = -\frac{\pi}{4} + k\pi$, $k \in \mathbb{Z}$.

$$\boxed{\begin{array}{c} \text{(a) } x = \frac{\pi}{3} + 2k\pi,\ \frac{2\pi}{3} + 2k\pi \\ \text{(b) } x = \frac{\pi}{6} + k\pi,\ \frac{5\pi}{6} + k\pi \\ \text{(c) } x = -\frac{\pi}{4} + k\pi \end{array}}$$

## Problem 21 :



**Answer:**

**(a)** From the left graph, $\sin x = 0.5$ has solutions $x = \frac{\pi}{6}, \frac{5\pi}{6}$ on $[0, 2\pi]$.

**(b)** Analytic: reference angle $\frac{\pi}{6}$. Sine positive in QI and QII: $\frac{\pi}{6}, \pi - \frac{\pi}{6} = \frac{5\pi}{6}$. Confirmed.

**(c)** From the right graph, $\cos x = -\frac{\sqrt{3}}{2}$ has solutions $x = \frac{5\pi}{6}, \frac{7\pi}{6}$ on $[0, 2\pi]$.

**(d)** Reference angle $\frac{\pi}{6}$. Cosine negative in QII and QIII: $\pi - \frac{\pi}{6} = \frac{5\pi}{6}$, $\pi + \frac{\pi}{6} = \frac{7\pi}{6}$.
General: $x = \frac{5\pi}{6} + 2k\pi$ or $x = \frac{7\pi}{6} + 2k\pi$, $k \in \mathbb{Z}$.

$$\boxed{\text{(a) } \frac{\pi}{6}, \frac{5\pi}{6} \text{ (c) } \frac{5\pi}{6}, \frac{7\pi}{6} \text{ (d) } x = \frac{5\pi}{6} + 2k\pi,\ \frac{7\pi}{6} + 2k\pi}$$

## Problem 22 :

**Answer:**

**(a)** $\cos(3x) = \cos(2x + x) = \cos(2x)\cos x - \sin(2x)\sin x$

$$\begin{aligned} &= (2\cos^2 x - 1)\cos x - 2\sin x\cos x\sin x \\ &= 2\cos^3 x - \cos x - 2\sin^2 x\cos x \\ &= 2\cos^3 x - \cos x - 2(1 - \cos^2 x)\cos x \end{aligned}$$

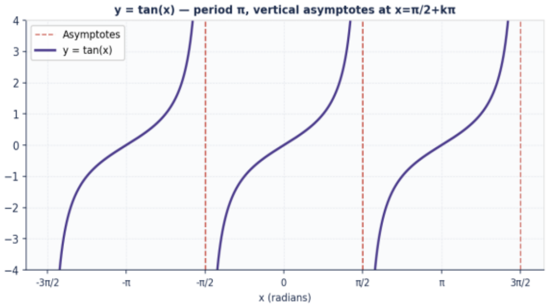
$= 2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x = 4\cos^3 x - 3\cos x$ ✓

**(b)** $\sin(3x) = \sin(2x + x) = \sin(2x)\cos x + \cos(2x)\sin x$

$$= 2\sin x\cos x\cos x + (1 - 2\sin^2 x)\sin x$$
$$= 2\sin x(1 - \sin^2 x) + \sin x - 2\sin^3 x$$

$= 2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x = 3\sin x - 4\sin^3 x$ ✓

## Problem 23 :



**Answer:**

**(a)** Period: $\pi$. Domain: $\mathbb{R} \setminus \{\frac{\pi}{2} + k\pi\}$. Range: $\mathbb{R}$.

**(b)** Vertical asymptotes at $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$ (visible on the graph).

**(c)** $\tan x$ is increasing on each open interval between consecutive asymptotes: $\left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right), \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$.

**(d)** $\tan x = 1$ at $x = \frac{\pi}{4}$ and $x = \frac{\pi}{4} - \pi = -\frac{3\pi}{4}$ on $[-3\pi/2, 3\pi/2]$.

$$\boxed{\begin{array}{c}\text{(a) Period } \pi\text{, domain } \mathbb{R} \setminus \left\{\frac{\pi}{2} + k\pi\right\}\text{, range } \mathbb{R} \\ \text{(b) } x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2} \\ \text{(c) inc on each interval} \\ \text{(d) } \frac{\pi}{4}, -\frac{3\pi}{4}\end{array}}$$

## Problem 24 :

**Answer:**

**(a)** $\sin(2x) = \sin x \Rightarrow 2\sin x\cos x = \sin x \Rightarrow \sin x(2\cos x - 1) = 0$

$\sin x = 0 \Rightarrow x = 0, \pi$

$\cos x = \frac{1}{2} \Rightarrow x = \frac{\pi}{3}, \frac{5\pi}{3}$

Solutions: $\left\{0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}\right\}$

**(b)** $\cos(2x) = 1 - \sin x \Rightarrow 1 - 2\sin^2 x = 1 - \sin x \Rightarrow -2\sin^2 x + \sin x = 0 \Rightarrow \sin x(1 - 2\sin x) = 0$

$\sin x = 0 \Rightarrow x = 0, \pi$

$\sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}$

Solutions: $\left\{0, \frac{\pi}{6}, \frac{5\pi}{6}, \pi\right\}$

**(c)** $2\sin^2 x - 3\sin x + 1 = 0 \Rightarrow (2\sin x - 1)(\sin x - 1) = 0$

$\sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}$

$\sin x = 1 \Rightarrow x = \frac{\pi}{2}$

Solutions: $\left\{\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}\right\}$

$$\boxed{\begin{array}{c}\text{(a) } 0, \frac{\pi}{3}, \pi, \frac{5\pi}{3} \\ \text{(b) } 0, \frac{\pi}{6}, \frac{5\pi}{6}, \pi \\ \text{(c) } \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}\end{array}}$$

## Problem 25 :

**Answer:**

**(a)** $R = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\tan\varphi = \frac{1}{1} = 1 \Rightarrow \varphi = \frac{\pi}{4}$

$$f(x) = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$$

**(b)** $R = \sqrt{3^2 + 4^2} = 5$

$\tan\varphi = \frac{4}{3} \Rightarrow \varphi = \arctan\left(\frac{4}{3}\right)$

$$g(x) = 5\sin\left(x + \arctan\frac{4}{3}\right)$$

**(c)** $R = \sqrt{1^2 + (-\sqrt{3})^2} = 2$

We want $R\sin(x + \varphi)$: $a = 1$, $b = -\sqrt{3}$

$\sin\varphi = \frac{b}{R} = -\frac{\sqrt{3}}{2}$, $\cos\varphi = \frac{a}{R} = \frac{1}{2} \Rightarrow \varphi = -\frac{\pi}{3}$

$$h(x) = 2\sin\left(x - \frac{\pi}{3}\right)$$

$$\boxed{\begin{array}{c}\text{(a) } \sqrt{2}\sin\left(x + \frac{\pi}{4}\right) \\ \text{(b) } 5\sin\left(x + \arctan\frac{4}{3}\right) \\ \text{(c) } 2\sin\left(x - \frac{\pi}{3}\right)\end{array}}$$

## Problem 26 :

**Answer:**

**(a)** $\cos(2\alpha) = 1 - 2\sin^2\alpha \Rightarrow \sin^2\alpha = \frac{1 - \cos(2\alpha)}{2}$

Replace $\alpha$ with $\frac{\theta}{2}$: $\sin^2\left(\frac{\theta}{2}\right) = \frac{1 - \cos\theta}{2}$

$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1 - \cos\theta}{2}}$ ✓

**(b)** $\theta = \frac{\pi}{4}$: $\sin\left(\frac{\pi}{8}\right) = \sqrt{\frac{1 - \cos(\pi/4)}{2}} = \sqrt{\frac{1 - \sqrt{2}/2}{2}} = \sqrt{\frac{2 - \sqrt{2}}{4}} = \frac{\sqrt{2 - \sqrt{2}}}{2}$

**(c)** $\theta = \frac{\pi}{6}$: $\cos\left(\frac{\pi}{12}\right) = \sqrt{\frac{1 + \cos(\pi/6)}{2}} = \sqrt{\frac{1 + \sqrt{3}/2}{2}} = \sqrt{\frac{2 + \sqrt{3}}{4}} = \frac{\sqrt{2 + \sqrt{3}}}{2}$

$$\boxed{\text{(b) } \frac{\sqrt{2 - \sqrt{2}}}{2} \text{(c) } \frac{\sqrt{2 + \sqrt{3}}}{2}}$$

## Problem 27 :

**Answer:**

**(a)** $\sin x > \frac{1}{2}$:
$\sin x = \frac{1}{2}$ at $x = \frac{\pi}{6}, \frac{5\pi}{6}$.
Solution: $\left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$

**(b)** $\cos x \le -\frac{\sqrt{2}}{2}$:
$\cos x = -\frac{\sqrt{2}}{2}$ at $x = \frac{3\pi}{4}, \frac{5\pi}{4}$.
Cosine is less than this in QIII:
Solution: $\left[\frac{3\pi}{4}, \frac{5\pi}{4}\right]$

**(c)** $2\sin^2 x - 1 > 0 \Rightarrow \sin^2 x > \frac{1}{2} \Rightarrow |\sin x| > \frac{1}{\sqrt{2}}$
$\sin x > \frac{1}{\sqrt{2}}$: $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$
$\sin x < -\frac{1}{\sqrt{2}}$: $\left(\frac{5\pi}{4}, \frac{7\pi}{4}\right)$
Union: $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right) \cup \left(\frac{5\pi}{4}, \frac{7\pi}{4}\right)$

$$\boxed{\text{(a) } (\frac{\pi}{6}, \frac{5\pi}{6}) \text{(b) } [\frac{3\pi}{4}, \frac{5\pi}{4}] \text{(c) } (\frac{\pi}{4}, \frac{3\pi}{4}) \cup (\frac{5\pi}{4}, \frac{7\pi}{4})}$$

## Problem 28 :

**Answer:**

**(a)** $a = c\sin\alpha = 10\sin\left(\frac{\pi}{5}\right)$, $b = c\cos\alpha = 10\cos\left(\frac{\pi}{5}\right)$

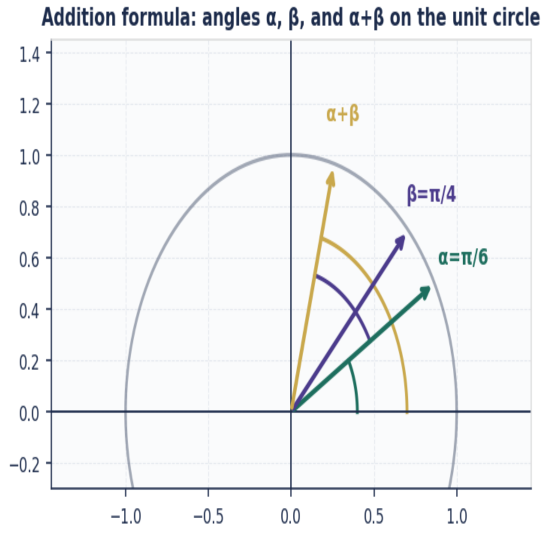
**(b)** $a^2 + b^2 = 100\sin^2(\pi/5) + 100\cos^2(\pi/5) = 100(\sin^2 + \cos^2) = 100 = c^2$ ✓

**(c)** $\beta = \frac{\pi}{2} - \frac{\pi}{5} = \frac{3\pi}{10} = 54^\circ$

**(d)** Area $= \frac{1}{2}ab = \frac{1}{2} \cdot 10\sin(\pi/5) \cdot 10\cos(\pi/5) = 50\sin(\pi/5)\cos(\pi/5) = 25\sin\left(\frac{2\pi}{5}\right)$

$$\boxed{\begin{array}{c}\text{(a) } a = 10\sin\frac{\pi}{5},\ b = 10\cos\frac{\pi}{5} \\ \text{(b) Verified ; (c) } \frac{3\pi}{10} = 54^\circ \\ \text{(d) } 25\sin\frac{2\pi}{5}\end{array}}$$

## Problem 29 :



**Answer:**

**(a)** $\alpha = \frac{\pi}{6}$, $\beta = \frac{\pi}{4}$, $\alpha + \beta = \frac{5\pi}{12} = 75^\circ$.

**(b)** Approximate coordinates: $(\cos 75^\circ, \sin 75^\circ) \approx (0.259, 0.966)$.

**(c)** $\cos(\pi/6 + \pi/4) = \cos(\pi/6)\cos(\pi/4) - \sin(\pi/6)\sin(\pi/4)$

$$= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} - \frac{1}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{6} - \sqrt{2}}{4}$$

**(d)** $\sin(\pi/6 + \pi/4) = \sin(\pi/6)\cos(\pi/4) + \cos(\pi/6)\sin(\pi/4)$

$$= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{2} + \sqrt{6}}{4}$$

Check: $\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)^2 + \left(\frac{\sqrt{6}+\sqrt{2}}{4}\right)^2 = \frac{(6-2\sqrt{12}+2)+(6+2\sqrt{12}+2)}{16} = \frac{16}{16} = 1$ ✓

## Problem 30 :

**Answer:**

**(a)** Rotation by $\beta$ maps $(\cos\alpha, \sin\alpha)$ to $(\cos(\alpha + \beta), \sin(\alpha + \beta))$.

**(b)** Verification for $\alpha = \beta = \pi/4$:

$$\sin(\pi/4 + \pi/4) = \sin(\pi/2) = 1$$

RHS: $\sin(\pi/4)\cos(\pi/4) + \cos(\pi/4)\sin(\pi/4) = 2\left(\frac{\sqrt{2}}{2}\right)^2 = 1$ ✓

**(c)** $\sin(\pi/2 - \theta) = \sin(\pi/2)\cos\theta - \cos(\pi/2)\sin\theta = \cos\theta$ ✓

**(d)** $\cos(\alpha + \beta) = \sin(\pi/2 - (\alpha + \beta)) = \sin((\pi/2 - \alpha) - \beta)$
$= \sin(\pi/2 - \alpha)\cos\beta - \cos(\pi/2 - \alpha)\sin\beta = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ ✓

## Problem 31 :

**Answer:**

**(a)** $A = 4$, $T = \frac{2\pi}{3}$, no phase shift.

**(b)** $A = 2$ (amplitude is absolute value), $T = \frac{2\pi}{1/2} = 4\pi$, vertical shift up 1, reflected.

**(c)** $A = 3$, $T = \frac{2\pi}{2} = \pi$, phase shift: $2x - \pi/4 = 0 \Rightarrow x = \frac{\pi}{8}$ (right).

$$\boxed{\begin{array}{c}\text{(a) } A = 4,\ T = \frac{2\pi}{3} \\ \text{(b) } A = 2,\ T = 4\pi \\ \text{(c) } A = 3,\ T = \pi, \text{ shift right } \frac{\pi}{8}\end{array}}$$

## Problem 32 :

**Answer:**

**(a)** $\sin(A + B) + \sin(A - B) = (\sin A\cos B + \cos A\sin B) + (\sin A\cos B - \cos A\sin B) = 2\sin A\cos B$ ✓

**(b)** $2\sin(5x)\cos(3x) = \sin(8x) + \sin(2x)$

**(c)** $\sin(5x) + \sin(3x) = 2\sin(4x)\cos(x)$

$$\boxed{\text{(b) } \sin(8x) + \sin(2x) \text{(c) } 2\sin(4x)\cos(x)}$$

## Problem 33 :

**Answer:**

**(a)** Square and add both equations:
$(\sin x + \cos y)^2 + (\cos x - \sin y)^2 = 1^2 + 1^2 = 2$

**(b)** Expand: $\sin^2 x + 2\sin x\cos y + \cos^2 y + \cos^2 x - 2\cos x\sin y + \sin^2 y = 2$

$$(\sin^2 x + \cos^2 x) + (\sin^2 y + \cos^2 y) + 2(\sin x\cos y - \cos x\sin y) = 2$$

$1 + 1 + 2\sin(x - y) = 2 \Rightarrow 2\sin(x - y) = 0 \Rightarrow \sin(x - y) = 0 \Rightarrow x - y = k\pi$

**(c)** Case 1: $x = y$. Then $\sin x + \cos x = 1 \Rightarrow \sqrt{2}\sin(x + \pi/4) = 1 \Rightarrow \sin(x + \pi/4) = \frac{1}{\sqrt{2}}$
$x + \pi/4 = \pi/4$ or $3\pi/4 \Rightarrow x = 0$ or $x = \pi/2$.
Solutions: $(0, 0)$, $(\pi/2, \pi/2)$.

Case 2: $x = y + \pi$. Then $\sin(y + \pi) + \cos y = 1 \Rightarrow -\sin y + \cos y = 1$
Solutions: $y = 0, x = \pi$ and $y = \pi/2, x = 3\pi/2$ etc.
Full solution set on $[0, 2\pi)^2$: $(0,0), (\pi/2, \pi/2), (\pi, 0), (3\pi/2, \pi/2)$ (and possibly more depending on branch).

$$\boxed{(0,0),\ (\pi/2, \pi/2),\ (\pi, 0),\ (3\pi/2, \pi/2)}$$

## Problem 34 :

**Answer:**

**(a)** $f'(x) = \cos x + 2\cos(2x)$

**(b)** $\cos(2x) = 2\cos^2 x - 1 \Rightarrow f'(x) = \cos x + 4\cos^2 x - 2$

**(c)** Let $u = \cos x$: $4u^2 + u - 2 = 0$

$$u = \frac{-1 \pm \sqrt{1 + 32}}{8} = \frac{-1 \pm \sqrt{33}}{8}$$

$u_1 = \frac{-1+\sqrt{33}}{8} \approx 0.593$, $u_2 = \frac{-1-\sqrt{33}}{8} \approx -0.843$
Critical points: $x = \arccos(0.593) \approx 0.936$ and $2\pi - 0.936 \approx 5.347$ for $u_1$;
$x = \arccos(-0.843) \approx 2.568$ and $2\pi - 2.568 \approx 3.715$ for $u_2$.

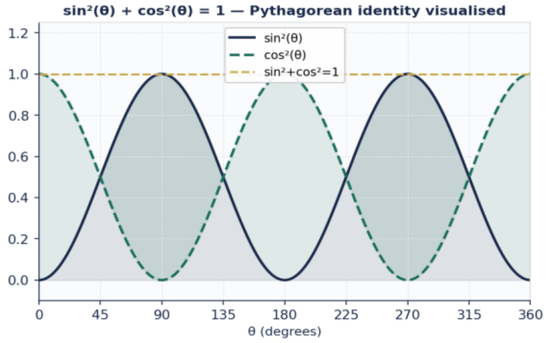
**(d)** Evaluate $f$ at critical points and endpoints:
$f(0) = 0, f(2\pi) = 0$

$$\begin{aligned} f(0.936) &= \sin(0.936) + \sin(1.872) \\ &\approx 0.806 + 0.952 \approx 1.758 \\ f(5.347) &\approx -0.758 \\ f(2.568) &\approx -0.758 \\ f(3.715) &\approx 1.758 \end{aligned}$$

Global maximum ≈ $1.758$ at $x \approx 0.936, 3.715$.

$$\boxed{\text{Global max } \approx 1.758 \text{ at } x \approx 0.936,\ 3.715}$$

## Problem 35 :



**Answer:**

**(a)** $\sin^2\theta = \cos^2\theta \Rightarrow \tan^2\theta = 1 \Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.

**(b)** At $\theta = 90^\circ = \frac{\pi}{2}$: $\sin^2 = 1$, $\cos^2 = 0$. Sum = 1.

**(c)** $(\sin^2\theta + \cos^2\theta)^2 = 1 \Rightarrow \sin^4\theta + 2\sin^2\theta\cos^2\theta + \cos^4\theta = 1$
$\sin^4\theta + \cos^4\theta = 1 - 2\sin^2\theta\cos^2\theta$ ✓

**(d)** $2\sin^2\theta\cos^2\theta = \frac{1}{2}\sin^2(2\theta) = \frac{1}{4}(1 - \cos(4\theta))$
$\sin^4\theta + \cos^4\theta = 1 - \frac{1}{4}(1 - \cos 4\theta) = \frac{3+\cos 4\theta}{4}$ ✓

# Problems 36–50 — Difficult & Real-World

## Problem 36 :

**Answer:**

**(a)** Drop altitude $h$ from $C$ to $AB$.
In the right triangle on the left: $h = b\sin A$.
In the right triangle on the right: $h = a\sin B$.

**(b)** From $h = b\sin A$ and $h = a\sin B$, we get $b\sin A = a\sin B \Rightarrow \frac{a}{\sin A} = \frac{b}{\sin B}$.

**(c)** Similarly, dropping an altitude from $A$ gives $\frac{b}{\sin B} = \frac{c}{\sin C}$.
So all three ratios are equal. ✓

**(d)** For $a = 8$, $A = 30^\circ$, $B = 45^\circ$:

$$\begin{aligned} b &= \frac{a\sin B}{\sin A} = \frac{8\sin 45^\circ}{\sin 30^\circ} = 8\sqrt{2} \\ C &= 180^\circ - 30^\circ - 45^\circ = 105^\circ \\ c &= \frac{a\sin C}{\sin A} = \frac{8\sin 105^\circ}{1/2} = 16\sin 105^\circ \\ &= 16 \cdot \frac{\sqrt{6} + \sqrt{2}}{4} = 4(\sqrt{6} + \sqrt{2}) \end{aligned}$$

$$\boxed{b = 8\sqrt{2},\ c = 4(\sqrt{6} + \sqrt{2})}$$

.

## Problem 37 :

**Answer:**

**(a)** Place triangle with $C$ at origin, $B$ on the positive x-axis:
$C = (0,0)$, $B = (a,0)$, $A = (b\cos C, b\sin C)$.

**(b)** $c^2 = \| AB \|^2 = (b\cos C - a)^2 + (b\sin C)^2$

**(c)** Expand: $= b^2\cos^2 C - 2ab\cos C + a^2 + b^2\sin^2 C$
$= a^2 + b^2(\sin^2 C + \cos^2 C) - 2ab\cos C = a^2 + b^2 - 2ab\cos C$ ✓

**(d)** $c^2 = 5^2 + 7^2 - 2(5)(7)\cos 60^\circ = 25 + 49 - 70(1/2) = 74 - 35 = 39$

$$c = \sqrt{39}$$

$$\boxed{c = \sqrt{39}}$$

## Problem 38 :

**Answer:**

**(a)** $\cos C = \frac{a^2+b^2-c^2}{2ab} = \frac{36+81-121}{2(6)(9)} = \frac{-4}{108} = -\frac{1}{27}$

$$C = \arccos\left(-\frac{1}{27}\right) \approx 92.1^\circ$$

**(b)** $\cos B = \frac{a^2+c^2-b^2}{2ac} = \frac{36+121-81}{2(6)(11)} = \frac{76}{132} = \frac{19}{33}$

$$B = \arccos\left(\frac{19}{33}\right) \approx 54.9^\circ$$

**(c)** $A = 180^\circ - B - C \approx 180^\circ - 92.1^\circ - 54.9^\circ = 33.0^\circ$

**(d)** Check Law of Sines: $a/\sin A = 6/\sin 33^\circ \approx 6/0.545 \approx 11.0$
$b/\sin B = 9/\sin 54.9^\circ \approx 9/0.817 \approx 11.0$ ✓

$$\boxed{A \approx 33.0^\circ,\ B \approx 54.9^\circ,\ C \approx 92.1^\circ}$$

## Problem 39 :

**Answer:**

**(a)** Let $P = \frac{A+B}{2}$, $Q = \frac{A-B}{2}$. Then $A = P + Q$, $B = P - Q$.

**(b)** $2\sin P\cos Q = 2\sin P\cos Q = \sin(P+Q) + \sin(P-Q)$ (using product-to-sum).

**(c)** $P + Q = A$, $P - Q = B$.
So $2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) = \sin A + \sin B$ ✓

**(d)** $\sin 75^\circ + \sin 15^\circ = 2\sin 45^\circ\cos 30^\circ = 2 \cdot \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{6}}{2}$

$$\boxed{\frac{\sqrt{6}}{2}}$$

## Problem 40 :

**Answer:**

**(a)** $\cos(2x) = 2\cos^2 x - 1$
$2\cos^2 x - 1 + \cos x = 0 \Rightarrow 2\cos^2 x + \cos x - 1 = 0$

**(b)** Factor: $(2\cos x - 1)(\cos x + 1) = 0$

**(c)** $\cos x = \frac{1}{2} \Rightarrow x = \frac{\pi}{3}, \frac{5\pi}{3}$
$\cos x = -1 \Rightarrow x = \pi$

**(d)** Solutions: $\left\{\frac{\pi}{3}, \pi, \frac{5\pi}{3}\right\}$

$$\boxed{\frac{\pi}{3},\ \pi,\ \frac{5\pi}{3}}$$

## Problem 41 :

**Answer:**

**(a)** $\sin(P+Q) - \sin(P-Q) = (\sin P\cos Q + \cos P\sin Q) - (\sin P\cos Q - \cos P\sin Q) = 2\cos P\sin Q$

**(b)** Let $P = \frac{\alpha+\beta}{2}$, $Q = \frac{\alpha-\beta}{2}$. Then $P + Q = \alpha$, $P - Q = \beta$.
So $\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$ ✓

**(c)** $\sin 80^\circ - \sin 40^\circ = 2\cos 60^\circ\sin 20^\circ = 2 \cdot \frac{1}{2}\sin 20^\circ = \sin 20^\circ$

$$\boxed{\sin 20^\circ}$$

## Problem 42 :

**Answer:**

**(a)** Let $\theta = \frac{\pi}{5}$. Then $5\theta = \pi \Rightarrow \cos(5\theta) = \cos\pi = -1$.

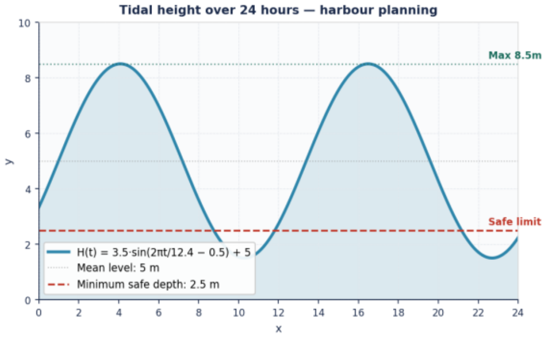
**(b)** $\cos(5\theta) = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$ (Chebyshev polynomial).
Let $c = \cos\theta$. Then $16c^5 - 20c^3 + 5c + 1 = 0$.

**(c)** Factor: $(c+1)(16c^4 - 16c^3 - 4c^2 + 4c + 1) = 0$.
$c = -1$ is extraneous (would give $\theta = \pi$).
The quartic factors as $(4c^2 - 2c - 1)(4c^2 + 2c - 1) = 0$.

**(d)** For $\theta = \pi/5$, $\cos\theta > 0$. The positive root of $4c^2 - 2c - 1 = 0$ is:
$c = \frac{2+\sqrt{4+16}}{8} = \frac{2+2\sqrt{5}}{8} = \frac{1+\sqrt{5}}{4}$.

$$\boxed{\cos\left(\frac{\pi}{5}\right) = \frac{1+\sqrt{5}}{4}}$$

## Problem 43 :



**Answer:**

**(a)** Amplitude = 3.5 m, Period = 12.4 hours, Mean height = 5 m.

**(b)** Maximum when $\frac{2\pi t}{12.4} - 0.5 = \frac{\pi}{2}$:
$t = \frac{12.4}{2\pi}\left(\frac{\pi}{2} + 0.5\right) \approx 4.08$ hours.

**(c)** $H(t) \geq 2.5 \Rightarrow 3.5\sin(\cdots) + 5 \geq 2.5 \Rightarrow \sin(\cdots) \geq -\frac{2.5}{3.5} = -\frac{5}{7}$
This is satisfied for most of the tidal cycle. Exact intervals require solving $\sin(\cdots) = -5/7$.

**(d)** $\frac{24}{12.4} \approx 1.94$ → approximately 2 complete cycles in 24 hours.

$$\boxed{\text{(a) 3.5, 12.4, 5(b) 4.08 h(c) Mostly all day(d) 2 cycles}}$$

## Problem 44 :

**Answer:**

**(a)** Let $h$ be the height and $d$ be the distance from B to the base.
$\tan 55^\circ = \frac{h}{d} \Rightarrow h = d\tan 55^\circ$
$\tan 35^\circ = \frac{h}{d+80} \Rightarrow h = (d+80)\tan 35^\circ$

**(b)** Equate: $d\tan 55^\circ = (d+80)\tan 35^\circ$
$d(\tan 55^\circ - \tan 35^\circ) = 80\tan 35^\circ$

**(c)** $d = \frac{80\tan 35^\circ}{\tan 55^\circ - \tan 35^\circ} \approx \frac{80(0.7002)}{1.4281 - 0.7002} \approx 76.97$ m
$h = d\tan 55^\circ \approx 76.97(1.4281) \approx 109.95$ m

**(d)** $h = \frac{80\tan 35^\circ \tan 55^\circ}{\tan 55^\circ - \tan 35^\circ} \approx 109.95$ m

$$\boxed{h \approx 109.95\text{ m}}$$

## Problem 45 :

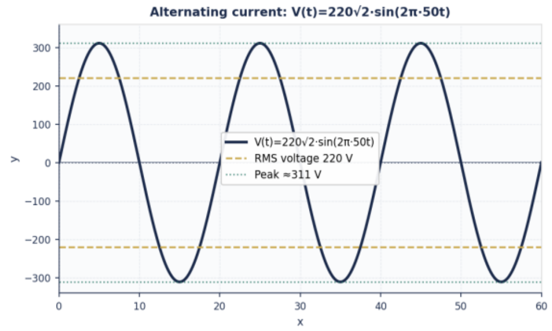
**Answer:**

**(a)** For $n = 1$: LHS = $1 + 2\cos x$.
RHS = $\frac{\sin(3x/2)}{\sin(x/2)} = \frac{\sin(x + x/2)}{\sin(x/2)} = \frac{\sin x\cos(x/2) + \cos x\sin(x/2)}{\sin(x/2)}$
$= \frac{2\sin(x/2)\cos(x/2)\cos(x/2) + \cos x\sin(x/2)}{\sin(x/2)} = 2\cos^2(x/2) + \cos x = 1 + \cos x + \cos x = 1 + 2\cos x$ ✓

**(b)** $x = \pi/3$: LHS = $1 + 2(1/2) = 2$. RHS = $\sin(\pi/2)/\sin(\pi/6) = 1/(1/2) = 2$. ✓

**(c)** For $n = 2$, $x = \pi/4$: LHS = $1 + 2(\sqrt{2}/2) + 0 = 1 + \sqrt{2}$.
RHS = $\sin(5\pi/8)/\sin(\pi/8) \approx 0.924/0.383 \approx 2.414 = 1 + \sqrt{2}$. ✓

## Problem 46 :



**Answer:**

**(a)** Peak = $220\sqrt{2} \approx 311$ V. Frequency = 50 Hz. Period = $1/50 = 0.02$ s = 20 ms.

**(b)** $V_{\text{rms}} = \frac{V_{\text{peak}}}{\sqrt{2}} = 220$ V.

**(c)** $220\sqrt{2}\sin(100\pi t) = 220 \Rightarrow \sin(100\pi t) = \frac{1}{\sqrt{2}}$
$100\pi t = \frac{\pi}{4}$ or $\frac{3\pi}{4} \Rightarrow t = \frac{1}{400}$ s = 2.5 ms, or $t = \frac{3}{400}$ s = 7.5 ms.

**(d)** Average power: $P_{\text{avg}} = \frac{V_{\text{rms}}^2}{R} = \frac{220^2}{100} = 484$ W.

$$\boxed{\begin{array}{c}\text{(a) 311 V, 50 Hz, 20 ms}\\ \text{(b) 220 V}\\ \text{(c) 2.5 ms, 7.5 ms}\\ \text{(d) 484 W}\end{array}}$$

## Problem 47 :

**Answer:**

**(a)** $\omega = \sqrt{\frac{9.8}{0.25}} = \sqrt{39.2} \approx 6.26$ rad/s.
$T = \frac{2\pi}{\omega} \approx 1.00$ s.

**(b)** $0.1\cos(6.26t) = 0.05 \Rightarrow \cos(6.26t) = 0.5 \Rightarrow 6.26t = \frac{\pi}{3} \Rightarrow t \approx 0.167$ s.

**(c)** $\theta'(t) = -\omega\theta_0\sin(\omega t)$. Max speed = $\omega\theta_0 = 6.26(0.1) = 0.626$ rad/s.

**(d)** $\theta'(t) = -\omega\theta_0\sin(\omega t)$, $\theta''(t) = -\omega^2\theta_0\cos(\omega t) = -\omega^2\theta(t)$.
So $\theta''(t) + \omega^2\theta(t) = 0$. ✓

$$\boxed{\begin{array}{c}\text{(a) 6.26 rad/s, 1.00 s}\\ \text{(b) 0.167 s}\\ \text{(c) 0.626 rad/s}\end{array}}$$

## Problem 48 :

**Answer:**

**(a)** $\cos^4\theta - \sin^4\theta = (\cos^2\theta - \sin^2\theta)(\cos^2\theta + \sin^2\theta)$

**(b)** $\cos^2\theta + \sin^2\theta = 1$

**(c)** $\cos^2\theta - \sin^2\theta = \cos(2\theta)$
So $\cos^4\theta - \sin^4\theta = \cos(2\theta)$ ✓

**(d)** This holds for all $\theta$, including cases where $\sin\theta = 0$ or $\cos\theta = 0$:
If $\sin\theta = 0$, LHS = 1 − 0 = 1, RHS = $\cos(0) = 1$.
If $\cos\theta = 0$, LHS = 0 − 1 = −1, RHS = $\cos(\pi) = -1$. ✓

$$\boxed{\cos^4\theta - \sin^4\theta = \cos(2\theta)}$$

## Problem 49 :

**Answer:**

**(a)** $x(t) = \sin(2t + \pi/2) = \cos(2t)$, $y(t) = \sin t$.

**(b)** $x = \cos(2t) = 1 - 2\sin^2 t = 1 - 2y^2$.

**(c)** $x = 1 - 2y^2$ is a parabola opening left. Vertex: $(1,0)$, axis of symmetry: the x-axis ($y = 0$).

**(d)** $(1,0)$: $x = 1 \Rightarrow 1 - 2y^2 = 1 \Rightarrow y = 0$. $\sin t = 0 \Rightarrow t = 0, \pi$.

**(e)** As $t$ goes from $0$ to $2\pi$, the curve starts at $(1,0)$, moves left to $(-1,1)$, back to $(1,0)$, left to $(-1,-1)$, and returns to $(1,0)$. It traces a figure-8 / parabolic loop.

## Problem 50 :

**Answer:**

**(a)** $H = R\sin\theta$, $\frac{L}{2} = R\cos\theta$.

**(b)** $\tan\theta = \frac{H}{L/2} = \frac{3}{6} = \frac{1}{2} \Rightarrow \theta = \arctan(0.5) \approx 26.57^\circ$.

**(c)** $R = \frac{H}{\sin\theta} = \frac{3}{0.4472} \approx 6.708$ m.

**(d)** Vertical force per rafter: $F_v = W \cdot R \cdot \sin\theta = W \cdot H = 2000 \cdot 3 = 6000$ N.

**(e)** By similar triangles: $\frac{d}{R} = \frac{H-h}{H} \Rightarrow d = R \cdot \frac{H-h}{H} = 6.708 \cdot \frac{1.5}{3} = 3.354$ m from the ridge.

$\boxed{\text{(b) } \theta \approx 26.57^\circ \text{(c) } 6.708 \text{ m(d) } 6000 \text{ N(e) } 3.354 \text{ m}}$

## CHAPTER 7 — ANALYTIC GEOMETRY & VECTORS

# SOLUTIONS TO PROBLEMS

## Problems 1–10 — Instant Recall

### Problem 1 :

**Answer:**

**(a)** $\overrightarrow{AB} = (7-2, -1-3) = (5, -4)$
$\| \overrightarrow{AB} \| = \sqrt{5^2 + (-4)^2} = \sqrt{25+16} = \sqrt{41}$

**(b)** $\overrightarrow{CD} = (2-(-4), 5-1) = (6, 4)$
$\| \overrightarrow{CD} \| = \sqrt{6^2 + 4^2} = \sqrt{36+16} = \sqrt{52} = 2\sqrt{13}$

**(c)** $\overrightarrow{EF} = (4-0, 4-(-3)) = (4, 7)$
$\| \overrightarrow{EF} \| = \sqrt{4^2 + 7^2} = \sqrt{16+49} = \sqrt{65}$

**(d)** $\overrightarrow{PQ} = (a+3-a, b-5-b) = (3, -5)$
$\| \overrightarrow{PQ} \| = \sqrt{3^2 + (-5)^2} = \sqrt{9+25} = \sqrt{34}$

$$\boxed{\begin{array}{c}\text{(a) } (5,-4),\ \sqrt{41} \\ \text{(b) } (6,4),\ 2\sqrt{13} \\ \text{(c) } (4,7),\ \sqrt{65} \\ \text{(d) } (3,-5),\ \sqrt{34}\end{array}}$$

### Problem 2 :

**Answer:**

**(a)** $u + v = (4 + (-2), 1 + 3) = (2, 4)$
$u - v = (4 - (-2), 1 - 3) = (6, -2)$
$3u - 2v = (3(4) - 2(-2), 3(1) - 2(3))$
$= (12 + 4, 3 - 6) = (16, -3)$

**(b)** $u + v = (0 + 3, -5 + 2) = (3, -3)$
$u - v = (0 - 3, -5 - 2) = (-3, -7)$
$3u - 2v = (0 - 6, -15 - 4) = (-6, -19)$

**(c)** $u + v = (1 + 1, 1 + (-1)) = (2, 0)$
$u - v = (1 - 1, 1 - (-1)) = (0, 2)$
$3u - 2v = (3 - 2, 3 - (-2)) = (1, 5)$

$$\boxed{\begin{array}{c}\text{(a) } (2,4), (6,-2), (16,-3) \\ \text{(b) } (3,-3), (-3,-7), (-6,-19) \\ \text{(c) } (2,0), (0,2), (1,5)\end{array}}$$

### Problem 3 :

**Answer:**

**(a)** $\boldsymbol{M} = \left(\frac{2+8}{2}, \frac{6+2}{2}\right) = (\mathbf{5}, \mathbf{4})$

**(b)** $\boldsymbol{M} = \left(\frac{-3+5}{2}, \frac{4+(-2)}{2}\right) = (\mathbf{1}, \mathbf{1})$

**(c)** $\boldsymbol{M} = \left(\frac{0+4}{2}, \frac{7+7}{2}\right) = (\mathbf{2}, \mathbf{7})$

**(d)** $\boldsymbol{M} = \left(\frac{a+c}{2}, \frac{b+d}{2}\right)$

$$\boxed{\textbf{(a) } (\mathbf{5}, \mathbf{4})\textbf{(b) } (\mathbf{1}, \mathbf{1})\textbf{(c) } (\mathbf{2}, \mathbf{7})\textbf{(d) } \left(\frac{\boldsymbol{a}+\boldsymbol{c}}{\mathbf{2}}, \frac{\boldsymbol{b}+\boldsymbol{d}}{\mathbf{2}}\right)}$$

### Problem 4 :

**Answer:**

**(a)** Slope $m = \frac{9-3}{4-1} = \frac{6}{3} = 2$
Using point-slope: $y - 3 = 2(x - 1) \Rightarrow y = 2x + 1$

**(b)** Slope $m = \frac{0-5}{3-(-2)} = \frac{-5}{5} = -1$
Using $D(3,0)$: $y - 0 = -1(x - 3) \Rightarrow y = -x + 3$

**(c)** Slope $m = \frac{-4-(-4)}{6-0} = 0 \Rightarrow$ horizontal line $y = -4$

**(d)** Both points have $x = 2 \Rightarrow$ vertical line $x = 2$

$$\boxed{\text{(a) } y = 2x + 1 \text{(b) } y = -x + 3 \text{(c) } y = -4 \text{(d) } x = 2}$$

### Problem 5 :

**Answer:**

**(a)** $u \cdot v = 3(-4) + 4(3) = -12 + 12 = 0 \rightarrow$ **Perpendicular**

**(b)** $u \cdot v = 2(3) + 1(-1) = 6 - 1 = 5 > 0 \rightarrow$ **Acute**

**(c)** $u \cdot v = 1(0) + 0(5) = 0 \rightarrow$ **Perpendicular**

**(d)** $u \cdot v = 2(4) + (-3)(6) = 8 - 18 = -10 < 0 \rightarrow$ **Obtuse**

$$\boxed{\text{(a) Perpendicular(b) Acute(c) Perpendicular(d) Obtuse}}$$

### Problem 6 :

**Answer:**

**(a)** Normal vector $n = (3, -4)$. Direction vector can be $(4, 3)$ (perpendicular to normal).

**(b)** Normal vector $n = (1, 2)$. Direction vector $(2, -1)$.

**(c)** Normal vector $n = (5, 0)$. Vertical line $x = 2$. Direction vector $(0, 1)$.

**(d)** Normal vector $n = (-2, 3)$. Direction vector $(3, 2)$.

$$\boxed{\text{(a) } n = (3,-4) \text{(b) } n = (1,2) \text{(c) } n = (5,0) \text{(d) } n = (-2,3)}$$

### Problem 7 :

**Answer:**

**(a)** Slopes: 3 and 3. Same slope → **Parallel**

**(b)** Slopes: 2 and $-\frac{1}{2}$. Product $= -1 \rightarrow$ **Perpendicular**

**(c)** First: $4x - 2y + 1 = 0 \Rightarrow y = 2x + \frac{1}{2}$. Slope $= 2$.
Second: $2x - y + 5 = 0 \Rightarrow y = 2x + 5$. Slope $= 2$.
Same slope → **Parallel**

**(d)** First: $x+3y=6 \Rightarrow y=-\frac{1}{3}x+2$. Slope $=-\frac{1}{3}$.
Second: $3x-y=4 \Rightarrow y=3x-4$. Slope $=3$.
Product $=-1 \rightarrow$ **Perpendicular**

$$\boxed{\text{(a) Parallel(b) Perpendicular(c) Parallel (d) Perpendicular}}$$

## Problem 8 :

**Answer:**

**(a)** $\overrightarrow{AB}=(2,4)$, $\overrightarrow{AC}=(4,8)$
Cross product: $2(8)-4(4)=16-16=0 \rightarrow$ **Collinear**

**(b)** $\overrightarrow{DE}=(2,3)$, $\overrightarrow{DF}=(4,5)$
Cross product: $2(5)-3(4)=10-12=-2 \neq 0 \rightarrow$ **Not collinear**

**(c)** $\overrightarrow{GH}=(3,-3)$, $\overrightarrow{GI}=(5,-5)$
Cross product: $3(-5)-(-3)(5)=-15+15=0 \rightarrow$ **Collinear**

$$\boxed{\text{(a) Collinear(b) Not collinear(c) Collinear}}$$

## Problem 9 :

**Answer:**

**(a)** $d=\frac{|3(1)-4(2)+10|}{\sqrt{3^2+(-4)^2}}=\frac{|3-8+10|}{5}=\frac{5}{5}=1$

**(b)** $d=\frac{|0+0-13|}{\sqrt{25+144}}=\frac{13}{13}=1$

**(c)** $d=\frac{|3+3-2|}{\sqrt{1+1}}=\frac{4}{\sqrt{2}}=2\sqrt{2}$

$$\boxed{\text{(a) }1\text{(b) }1\text{(c) }2\sqrt{2}}$$

## Problem 10 :

**Answer:**

**(a)** $2x+1=-x+7 \Rightarrow 3x=6 \Rightarrow x=2$, $y=2(2)+1=5$
Point: $(2,5)$

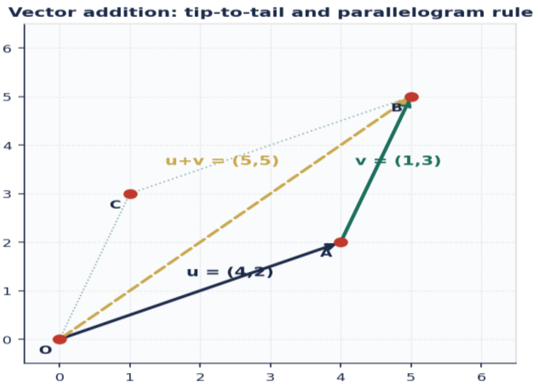
**(b)** From $x-y=1$, $x=y+1$. Substitute: $3(y+1)+2y=12 \Rightarrow 5y+3=12 \Rightarrow y=\frac{9}{5}$, $x=\frac{14}{5}$
Point: $\left(\frac{14}{5},\frac{9}{5}\right)$

**(c)** $y=3$, $2x-3=5 \Rightarrow 2x=8 \Rightarrow x=4$
Point: $(4,3)$

$$\boxed{\text{(a) }(2,5)\text{(b) }\left(\frac{14}{5},\frac{9}{5}\right)\text{(c) }(4,3)}$$

## Problems 11–35 — Medium

## Problem 11 :



**Answer:**

**(a)** From the graph: $u=(4,2)$, $v=(1,3)$. $u+v=(5,5)$.

**(b)** The parallelogram rule shows the diagonal $OB=(5,5)=u+v$. Confirmed.

**(c)** $\parallel u\parallel=\sqrt{4^2+2^2}=\sqrt{20}=2\sqrt{5}$
$\parallel v\parallel=\sqrt{1^2+3^2}=\sqrt{10}$
$\parallel u+v\parallel=\sqrt{5^2+5^2}=\sqrt{50}=5\sqrt{2}\approx 7.07$
$\parallel u\parallel+\parallel v\parallel=4.47+3.16=7.63>7.07$ ✓
Triangle inequality holds.

**(d)** Unit vector: $\frac{(5,5)}{5\sqrt{2}}=\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$

$$\boxed{\text{(a) }(5,5)\text{(c) }2\sqrt{5},\ \sqrt{10},\ 5\sqrt{2}\text{(d) }\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)}$$

## Problem 12 :

**Answer:**

**(a)** $\overrightarrow{AB}=(4-1,3-(-1))=(3,4)$
$\overrightarrow{DC}=C-D=(7-4,-1-(-5))=(3,4)$
Equal.

**(b)** Since $\overrightarrow{AB}=\overrightarrow{DC}$, one pair of opposite sides are equal and parallel.
ABCD is a parallelogram.

**(c)** Midpoint of $AC$: $\left(\frac{1+7}{2},\frac{-1+(-1)}{2}\right)=(4,-1)$
Midpoint of $BD$: $\left(\frac{4+4}{2},\frac{3+(-5)}{2}\right)=(4,-1)$
Same midpoint $\rightarrow$ diagonals bisect each other.
**(d)** ABCD is a parallelogram.

$$\boxed{\begin{array}{c}\text{(a) }(3,4),(3,4)\\ \text{(b) Parallelogram}\\ \text{(c) Same midpoint }(4,-1)\\ \text{(d) Parallelogram}\end{array}}$$

## Problem 13 :

**Answer:**

**(a)** $y+2=4(x-3) \Rightarrow y=4x-14$

**(b)** Slope of given line $=3$. Perpendicular slope $=-\frac{1}{3}$.
$y-5=-\frac{1}{3}(x+1) \Rightarrow y=-\frac{x}{3}+\frac{14}{3}$

**(c)** $2x - 3y + 6 = 0 \Rightarrow y = \frac{2}{3}x + 2$. Slope = $\frac{2}{3}$.
Through $C(4{,}0)$: $y = \frac{2}{3}(x-4) \Rightarrow y = \frac{2x}{3} - \frac{8}{3}$

**(d)** Midpoint of $DE$: $M = \left(\frac{1+5}{2}, \frac{3+1}{2}\right) = (3{,}2)$
Slope of $DE = \frac{1-3}{5-1} = -\frac{1}{2}$
Perpendicular slope = 2
Line: $y - 2 = 2(x-3) \Rightarrow y = 2x - 4$

$$\boxed{\begin{array}{c}\text{(a) } y = 4x - 14 \\ \text{(b) } y = -\dfrac{x}{3} + \dfrac{14}{3} \\ \text{(c) } y = \dfrac{2x}{3} - \dfrac{8}{3} \\ \text{(d) } y = 2x - 4\end{array}}$$

## Problem 14 :

**Answer:**

**(a)** $AB = \sqrt{(6-0)^2 + 0^2} = 6$
$BC = \sqrt{(2-6)^2 + (4-0)^2} = \sqrt{16+16} = 4\sqrt{2}$
$AC = \sqrt{(2-0)^2 + (4-0)^2} = \sqrt{4+16} = 2\sqrt{5}$

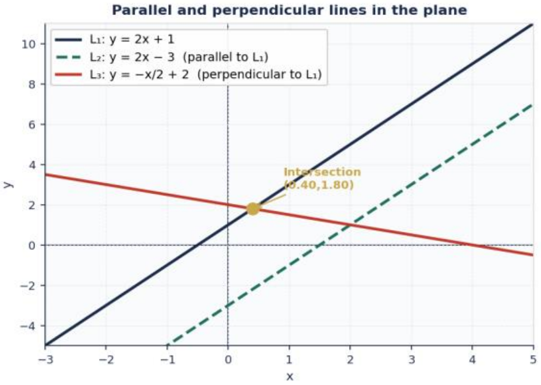
**(b)** Perimeter = $6 + 4\sqrt{2} + 2\sqrt{5}$

**(c)** $M_a$ (midpoint of $BC$) = $\left(\frac{6+2}{2}, \frac{0+4}{2}\right) = (4{,}2)$
$M_b$ (midpoint of $AC$) = $\left(\frac{0+2}{2}, \frac{0+4}{2}\right) = (1{,}2)$
$M_c$ (midpoint of $AB$) = $\left(\frac{0+6}{2}, \frac{0+0}{2}\right) = (3{,}0)$

**(d)** Median from $A(0{,}0)$ to $M_a(4{,}2)$: slope = $\frac{2}{4} = \frac{1}{2}$
Equation: $y = \frac{1}{2}x$

$$\boxed{\begin{array}{c}\text{(a) } 6,\ 4\sqrt{2},\ 2\sqrt{5} \\ \text{(b) } 6 + 4\sqrt{2} + 2\sqrt{5} \\ \text{(c) } (4{,}2), (1{,}2), (3{,}0) \\ \text{(d) } y = \dfrac{1}{2}x\end{array}}$$

## Problem 15 :



**Answer:**

**(a)** From the graph: $L_1: y = 2x + 1$, $L_2: y = 2x - 3$, $L_3: y = -\frac{x}{2} + 2$.

**(b)** $L_1$ slope = 2, $L_2$ slope = 2. Same slope → parallel ✓

**(c)** $L_1$ slope = 2, $L_3$ slope = $-\frac{1}{2}$. Product = $-1$ → perpendicular ✓

**(d)** Intersection of $L_1$ and $L_3$:
$2x + 1 = -\frac{x}{2} + 2 \Rightarrow 2x + \frac{x}{2} = 1 \Rightarrow \frac{5x}{2} = 1 \Rightarrow x = \frac{2}{5}$

$$y = 2\left(\frac{2}{5}\right) + 1 = \frac{4}{5} + 1 = \frac{9}{5}$$

Point: $\left(\frac{2}{5}, \frac{9}{5}\right)$

$$\boxed{\text{(a) } y = 2x + 1,\ y = 2x - 3,\ y = -\frac{x}{2} + 2 \text{(d) } \left(\frac{2}{5}, \frac{9}{5}\right)}$$

## Problem 16 :

**Answer:**

**(a)** $u \cdot v = 0$, $\| u \| = 1$, $\| v \| = 1$. $\cos\theta = 0 \Rightarrow \theta = 90^\circ$

**(b)** $u \cdot v = 3$, $\| u \| = 3\sqrt{2}$, $\| v \| = 1$. $\cos\theta = \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}} \Rightarrow \theta = 45^\circ$

**(c)** $u \cdot v = -2 + 2 = 0 \Rightarrow \theta = 90^\circ$

**(d)** $u \cdot v = 10 - 10 = 0 \Rightarrow \theta = 90^\circ$

$$\boxed{\text{(a) } 90^\circ \text{(b) } 45^\circ \text{(c) } 90^\circ \text{(d) } 90^\circ}$$

## Problem 17 :

**Answer:**

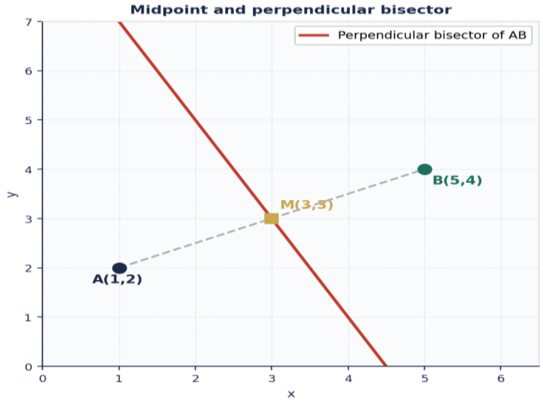
**(a)** Let rectangle vertices be $O(0{,}0)$, $A(a{,}0)$, $B(a,b)$, $C(0,b)$.
Diagonals: $OB = (a,b)$, $AC = (-a,b)$.

**(b)** $\| OB \| = \sqrt{a^2 + b^2}$, $\| AC \| = \sqrt{(-a)^2 + b^2} = \sqrt{a^2 + b^2}$.

**(c)** Equal ✓

**(d)** Midpoint of $OB$: $\left(\frac{a}{2}, \frac{b}{2}\right)$.
Midpoint of $AC$: $\left(\frac{a+0}{2}, \frac{0+b}{2}\right) = \left(\frac{a}{2}, \frac{b}{2}\right)$.
Same midpoint → diagonals bisect each other.

## Problem 18 :



**Answer:**

**(a)** $A(1{,}2)$, $B(5{,}4)$, $M(3{,}3)$.
Check: $\left(\frac{1+5}{2}, \frac{2+4}{2}\right) = (3{,}3)$ ✓

**(b)** Slope of $AB = \frac{4-2}{5-1} = \frac{1}{2}$.

Perpendicular slope = $-2$. Read from graph: ✓

**(c)** Perpendicular bisector through $M(3,3)$ with slope $-2$:
$y - 3 = -2(x-3) \Rightarrow y = -2x + 9$

**(d)** $P(1,5)$:

$$PA = \sqrt{(1-1)^2 + (5-2)^2} = 3$$

$$PB = \sqrt{(1-5)^2 + (5-4)^2} = \sqrt{16+1} = \sqrt{17}$$

Not equal → P is not on the perpendicular bisector.

Check: $-2(1) + 9 = 7 \neq 5$. ✓

(a) $A(1,2), B(5,4), M(3,3)$
(c) $y = -2x + 9$
(d) $P$ not on bisector

## Problem 19 :

**Answer:**

**(a)** $u \cdot v = 3(2) + (-1)(k) = 6 - k = 0 \Rightarrow k = 6$

**(b)** Parallel: $3k - (-1)(2) = 0 \Rightarrow 3k + 2 = 0 \Rightarrow k = -\frac{2}{3}$

**(c)** $\cos 45^\circ = \frac{6-k}{\sqrt{10}\sqrt{4+k^2}}$

$$\frac{1}{\sqrt{2}} = \frac{6-k}{\sqrt{10(4+k^2)}}$$

Square: $\frac{1}{2} = \frac{(6-k)^2}{10(4+k^2)}$

$$10(4+k^2) = 2(6-k)^2$$

$$40 + 10k^2 = 2(36 - 12k + k^2)$$

$$40 + 10k^2 = 72 - 24k + 2k^2$$

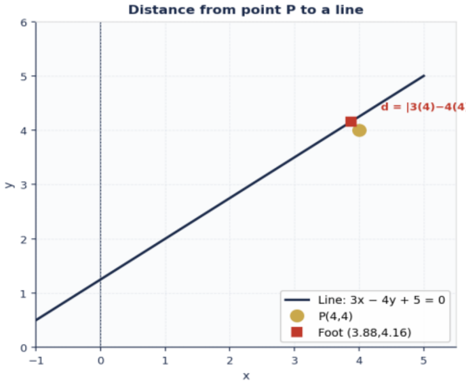
$8k^2 + 24k - 32 = 0 \Rightarrow k^2 + 3k - 4 = 0 \Rightarrow (k+4)(k-1) = 0$

$k = 1$ or $k = -4$

**(d)** $6 - k = 10 \Rightarrow k = -4$

(a) 6(b) $-\frac{2}{3}$(c) 1 or $-4$(d) $-4$

## Problem 20 :



**Answer:**

**(a)** From graph: $P(4,4)$, line $3x - 4y + 5 = 0$.

**(b)** $d = \frac{|3(4)-4(4)+5|}{\sqrt{3^2+(-4)^2}} = \frac{|12-16+5|}{5} = \frac{1}{5} = 0.2$

**(c)** $val = 3(4) - 4(4) + 5 = 1$

Foot: $x_f = 4 - \frac{3(1)}{25} = 4 - \frac{3}{25} = \frac{97}{25}$

$$y_f = 4 + \frac{4(1)}{25} = 4 + \frac{4}{25} = \frac{104}{25}$$

**(d)** Perpendicular to line through $P$: slope of line $= \frac{3}{4}$, so perpendicular slope = $-\frac{4}{3}$.

Equation: $y - 4 = -\frac{4}{3}(x-4) \Rightarrow y = -\frac{4x}{3} + \frac{28}{3}$

(b) 0.2(c) $\left(\frac{97}{25}, \frac{104}{25}\right)$ (d) $y = -\frac{4x}{3} + \frac{28}{3}$

## Problem 21 :

**Answer:**

**(a)** $d = \frac{|4(2)+3(3)-5|}{\sqrt{4^2+3^2}} = \frac{|8+9-5|}{5} = \frac{12}{5}$

**(b)** $val = 4(2) + 3(3) - 5 = 12$

Foot: $x_f = 2 - \frac{4(12)}{25} = 2 - \frac{48}{25} = \frac{2}{25}$

$$y_f = 3 - \frac{3(12)}{25} = 3 - \frac{36}{25} = \frac{39}{25}$$

**(c)** $A' = 2H - A$

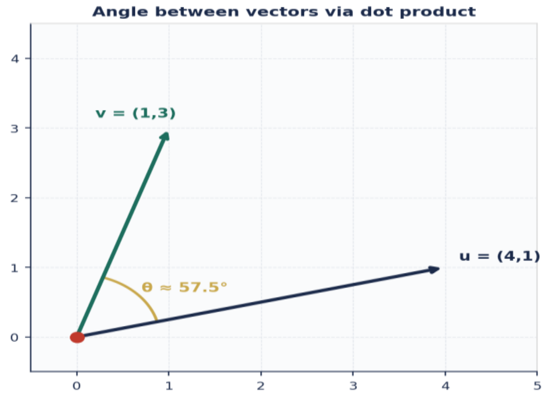
$$x' = 2\left(\frac{2}{25}\right) - 2 = \frac{4}{25} - \frac{50}{25} = -\frac{46}{25}$$

$$y' = 2\left(\frac{39}{25}\right) - 3 = \frac{78}{25} - \frac{75}{25} = \frac{3}{25}$$

**(d)** $d(A', L) = \frac{|4(-46/25)+3(3/25)-5|}{5} = \frac{|-184/25+9/25-125/25|}{5} = \frac{300}{25} \cdot \frac{1}{5} = \frac{12}{5}$. Same distance ✓

(a) $\frac{12}{5}$(c) $\left(-\frac{46}{25}, \frac{3}{25}\right)$

## Problem 22 :



**Answer:**

**(a)** $u = (4,1)$, $v = (1,3)$. $u \cdot v = 4 + 3 = 7$

**(b)** $\| u \| = \sqrt{17}$, $\| v \| = \sqrt{10}$

$\cos\theta = \frac{7}{\sqrt{170}} \Rightarrow \theta = \arccos\left(\frac{7}{\sqrt{170}}\right) \approx 57.5^\circ$

**(c)** Graph shows angle slightly less than $60^\circ$: consistent.

**(d)** Perpendicular to $u = (4,1)$: $w = (-1,4)$.
$\| w \| = \sqrt{1+16} = \sqrt{17} = \| u \|$.
$u \cdot w = -4 + 4 = 0$ ✓

$$\boxed{\text{(a) } 7 \text{(b) } \approx 57.5^\circ \text{(d) } w=(-1,4)}$$

## Problem 23 :

**Answer:**

**(a)** Slope of $BC = \frac{6-0}{3-8} = -\frac{6}{5}$. Altitude from $A$ has slope $\frac{5}{6}$.
Equation: $y = \frac{5}{6}x$

**(b)** Slope of $AC = \frac{6-0}{3-0} = 2$. Altitude from $B$ has slope $-\frac{1}{2}$.
Equation: $y = -\frac{1}{2}(x-8) \Rightarrow y = -\frac{x}{2} + 4$

**(c)** Intersection: $\frac{5x}{6} = -\frac{x}{2} + 4 \Rightarrow \frac{5x}{6} + \frac{3x}{6} = 4 \Rightarrow \frac{8x}{6} = 4 \Rightarrow x = 3$

$$y = \frac{5}{6}(3) = \frac{5}{2}$$
$$H = (3, \frac{5}{2})$$

**(d)** $AB$ is horizontal ($y = 0$). Altitude from $C$ is vertical through $C(3,6)$: $x = 3$.
$H(3, \frac{5}{2})$ has $x = 3$ ✓

$$\boxed{\text{(a) } y = \frac{5x}{6} \text{(b) } y = -\frac{x}{2} + 4 \text{(c) } \left(3, \frac{5}{2}\right)}$$

## Problem 24 :

**Answer:**

**(a)** $\overrightarrow{AB} = (3,6)$. For collinearity, $\overrightarrow{AC} = (\lambda - 1, 12)$ must be parallel to $\overrightarrow{AB}$.
Cross product: $3(12) - 6(\lambda - 1) = 0 \Rightarrow 36 - 6\lambda + 6 = 0 \Rightarrow 42 - 6\lambda = 0 \Rightarrow \lambda = 7$

**(b)** Slope $AB = \frac{6}{3} = 2$. Slope $AC = \frac{14-2}{\lambda-1} = \frac{12}{\lambda-1}$. Set equal: $\frac{12}{\lambda-1} = 2 \Rightarrow \lambda - 1 = 6 \Rightarrow \lambda = 7$

**(c)** $\lambda = 7$

**(d)** Through $A(1,2)$, slope 2: $y - 2 = 2(x - 1) \Rightarrow y = 2x$

$$\boxed{\lambda = 7,\ y = 2x}$$

## Problem 25 :

**Answer:**

**(a)** $\overrightarrow{AB} = (3,4)$, $\overrightarrow{AC} = (-4,3)$, $\overrightarrow{BC} = (-7,-1)$

**(b)** $\overrightarrow{AB} \cdot \overrightarrow{AC} = 3(-4) + 4(3) = -12 + 12 = 0$
$\overrightarrow{AB} \cdot \overrightarrow{BC} = 3(-7) + 4(-1) = -21 - 4 = -25$
$\overrightarrow{AC} \cdot \overrightarrow{BC} = (-4)(-7) + 3(-1) = 28 - 3 = 25$

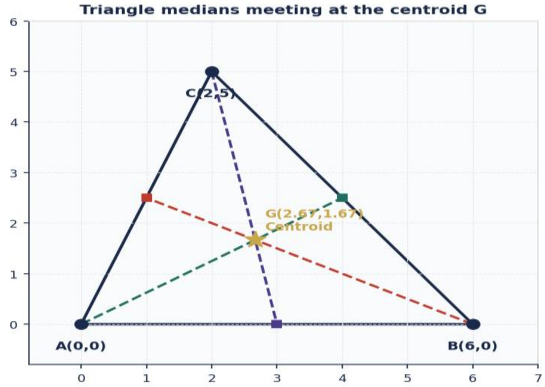
**(c)** $\overrightarrow{AB} \perp \overrightarrow{AC}$, so right angle at $A$.

**(d)** Area $= \frac{1}{2} \parallel \overrightarrow{AB} \parallel\parallel \overrightarrow{AC} \parallel = \frac{1}{2}(5)(5) = \frac{25}{2}$

$$\boxed{\text{Right angle at } A\text{, area } = \frac{25}{2}}$$

.

## Problem 26 :



**Answer:**

**(a)** $A(0,0)$, $B(6,0)$, $C(2,5)$.

$$G = \left(\frac{0+6+2}{3}, \frac{0+0+5}{3}\right) = \left(\frac{8}{3}, \frac{5}{3}\right)$$

**(b)** $M_a = \left(\frac{6+2}{2}, \frac{0+5}{2}\right) = (4, \frac{5}{2})$
Line from $A(0,0)$ to $M_a(4, \frac{5}{2})$: parametric $x = 4t, y = \frac{5t}{2}$.
At $t = \frac{2}{3}$: $x = \frac{8}{3}$, $y = \frac{5}{3} = G$ ✓

**(c)** $AG = \sqrt{\left(\frac{8}{3}\right)^2 + \left(\frac{5}{3}\right)^2} = \frac{\sqrt{89}}{3}$

$$GM_a = \sqrt{\left(4 - \frac{8}{3}\right)^2 + \left(\frac{5}{2} - \frac{5}{3}\right)^2} = \sqrt{\left(\frac{4}{3}\right)^2 + \left(\frac{5}{6}\right)^2}$$
$$= \frac{\sqrt{89}}{6}$$

Ratio $AG : GM_a = 2:1$ ✓

**(d)** $OG = \left(\frac{8}{3}, \frac{5}{3}\right) = \frac{OA+OB+OC}{3}$ ✓

## Problem 27 :

**Answer:**

**(a)** $x^2 + y^2 = 25$

**(b)** $(x-3)^2 + (y+2)^2 = 16$

**(c)** Centre $M = \left(\frac{1+7}{2}, \frac{1+5}{2}\right) = (4,3)$
Radius $r = \frac{1}{2} \parallel AB \parallel = \frac{1}{2}\sqrt{(6)^2 + (4)^2} = \frac{1}{2}\sqrt{52} = \sqrt{13}$
Equation: $(x-4)^2 + (y-3)^2 = 13$

**(d)** Radius $r = \parallel CP \parallel = \sqrt{(5-2)^2 + (7-3)^2} = \sqrt{9+16} = 5$
Equation: $(x-2)^2 + (y-3)^2 = 25$

$$\boxed{\begin{array}{c} \text{(a) } x^2 + y^2 = 25 \\ \text{(b) } (x-3)^2 + (y+2)^2 = 16 \\ \text{(c) } (x-4)^2 + (y-3)^2 = 13 \\ \text{(d) } (x-2)^2 + (y-3)^2 = 25 \end{array}}$$

## Problem 28 :

**Answer:**

**(a)** $x^2 - 6x + y^2 + 4y - 3 = 0$
$(x-3)^2 - 9 + (y+2)^2 - 4 - 3 = 0 \Rightarrow (x-3)^2 +$

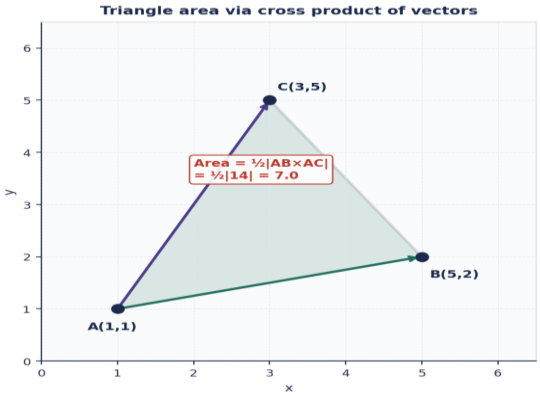
$(y+2)^2 = 16$
Centre: $(3,-2)$, radius 4

**(b)** $x^2+2x+y^2-8y+13=0$
$(x+1)^2-1+(y-4)^2-16+13=0 \Rightarrow (x+1)^2+(y-4)^2=4$
Centre: $(-1,4)$, radius 2

**(c)** $x^2-10x+y^2=0$
$(x-5)^2-25+y^2=0 \Rightarrow (x-5)^2+y^2=25$
Centre: $(5,0)$, radius 5

$$\boxed{\text{(a) } (3,-2),4 \text{(b) } (-1,4),2 \text{(c) } (5,0),5}$$

## Problem 29 :



**Answer:**

**(a)** $A(1,1)$, $B(5,2)$, $C(3,5)$.
$\overrightarrow{AB}=(4,1)$, $\overrightarrow{AC}=(2,4)$

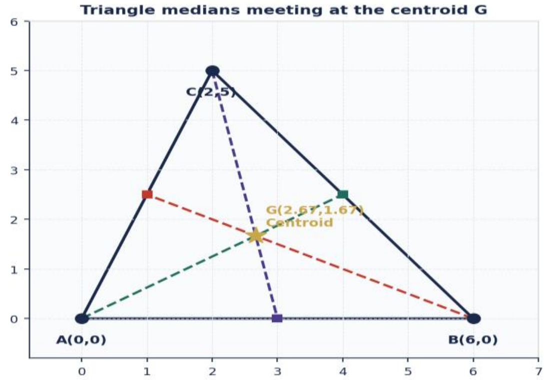
**(b)** Area $=\frac{1}{2}\mid 4(4)-1(2)\mid=\frac{1}{2}\mid 16-2\mid=7$

**(c)** Parallelogram area $=\mid 16-2\mid=14$

**(d)** $D=A+\overrightarrow{AB}+\overrightarrow{AC}=(1+4+2,1+1+4)=(7,6)$
Or $D=B+C-A=(5+3-1,2+5-1)=(7,6)$

$$\boxed{\text{(a) } AB=(4,1), AC=(2,4) \text{(b) } 7 \text{(c) } 14 \text{(d) } (7,6)}$$

## Problem 30 :



**Answer:**

**(a)** Define $G=\frac{A+B+C}{3}$. Show $G$ lies on all three medians.

**(b)** Median from $A$: midpoint of $BC=\frac{B+C}{2}$.
$A+\frac{2}{3}\left(\frac{B+C}{2}-A\right)=A+\frac{B+C}{3}-\frac{2A}{3}=\frac{A+B+C}{3}=G$ ✓

**(c)** Median from $B$: midpoint of $AC=\frac{A+C}{2}$.
$B+\frac{2}{3}\left(\frac{A+C}{2}-B\right)=\frac{A+B+C}{3}=G$ ✓

**(d)** $G$ divides each median in ratio $2:1$ from the vertex.

## Problem 31 :

**Answer:**

**(a)** Substitute $y=mx+3$: $x^2+(mx+3)^2=4 \Rightarrow (1+m^2)x^2+6mx+5=0$

**(b)** Tangency: discriminant = 0.
$$\Delta=(6m)^2-4(1+m^2)(5)=36m^2-20-20m^2 = 16m^2-20=0$$

**(c)** $16m^2=20 \Rightarrow m^2=\frac{5}{4} \Rightarrow m=\pm\frac{\sqrt{5}}{2}$

**(d)** Lines: $y=\pm\frac{\sqrt{5}}{2}x+3$
For $m=\frac{\sqrt{5}}{2}$: $x=-\frac{6m}{2(1+m^2)}=-\frac{3\sqrt{5}/2}{9/4}=-\frac{2\sqrt{5}}{3}$
$$y=mx+3=\frac{\sqrt{5}}{2}\left(-\frac{2\sqrt{5}}{3}\right)+3=-\frac{5}{3}+3=\frac{4}{3}$$
Tangent point: $\left(-\frac{2\sqrt{5}}{3},\frac{4}{3}\right)$

$$\boxed{m=\pm\frac{\sqrt{5}}{2}}$$

## Problem 32 :

**Answer:**

**(a)** $d=\frac{|3(-1)-4(2)+12|}{\sqrt{3^2+(-4)^2}}=\frac{|-3-8+12|}{5}=\frac{1}{5}$

**(b)** $val=3(-1)-4(2)+12=1$
$$H=(-1,2)-\frac{1}{25}(3,-4)=\left(-1-\frac{3}{25},2+\frac{4}{25}\right) =\left(-\frac{28}{25},\frac{54}{25}\right)$$

**(c)** $A'=2H-A=\left(-\frac{56}{25}+1,\frac{108}{25}-2\right)=\left(-\frac{31}{25},\frac{58}{25}\right)$

**(d)** Parallel to $L$: same normal $(3,-4)$ through $A$:
$3(x+1)-4(y-2)=0 \Rightarrow 3x+3-4y+8=0 \Rightarrow 3x-4y+11=0$

$$\boxed{\text{(a) } \frac{1}{5} \text{(b) } \left(-\frac{28}{25},\frac{54}{25}\right) \text{(c) } \left(-\frac{31}{25},\frac{58}{25}\right) \text{(d) } 3x-4y+11=0}$$

.

## Problem 33 :

**Answer:**

**(a)** $\parallel MA \parallel^2=\parallel MB \parallel^2$: $(x-1)^2+y^2=(x-5)^2+(y-4)^2$

**(b)** Expand: $x^2-2x+1+y^2=x^2-10x+25+y^2-8y+16$
$-2x+1=-10x+25-8y+16 \Rightarrow 8x+8y=40 \Rightarrow x+y=5$

**(c)** A straight line (perpendicular bisector of $AB$).

**(d)** Midpoint of $AB$ = $(3,2)$. Check: $3+2=5$ ✓

$$\boxed{x+y=5}$$

## Problem 34 :

**Answer:**

**(a)** Slope $=\frac{1-5}{6-2}=-1$.
Equation: $y-5=-1(x-2)\Rightarrow y=-x+7\Rightarrow x+y-7=0$

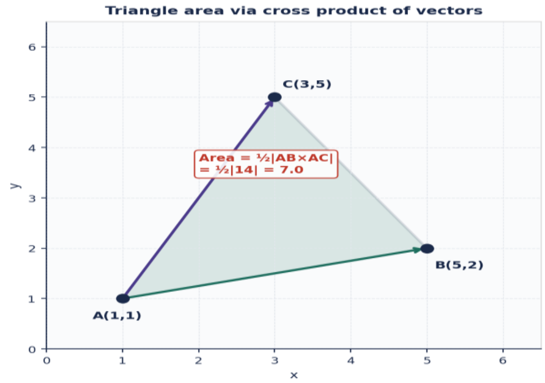
**(b)** $d=\frac{|0+0-7|}{\sqrt{1+1}}=\frac{7}{\sqrt{2}}=\frac{7\sqrt{2}}{2}$

**(c)** Perpendicular slope = 1 through $O(0,0)$: $y=x$

**(d)** Intersection of $y=x$ and $x+y=7$: $2x=7\Rightarrow x=\frac{7}{2}$, $y=\frac{7}{2}$
Foot: $\left(\frac{7}{2},\frac{7}{2}\right)$

$$\boxed{\text{(a) } x+y-7=0 \text{(b) } \frac{7\sqrt{2}}{2} \text{(c) } y=x \text{(d) } \left(\frac{7}{2},\frac{7}{2}\right)}$$

## Problem 35 :



**Answer:**

**(a)** $A(1,1)$, $B(5,2)$, $C(3,5)$.
$\overrightarrow{AB}=(4,1)$, $\overrightarrow{AC}=(2,4)$.
Area $=\frac{1}{2}\mid 4(4)-1(2)\mid=\frac{1}{2}(14)=7$

**(b)** Perpendicular to $AB=(4,1)$: direction $(-1,4)$, magnitude $\sqrt{17}$.
Vector of length 3: $v=\frac{3}{\sqrt{17}}(-1,4)=\left(-\frac{3}{\sqrt{17}},\frac{12}{\sqrt{17}}\right)$

**(c)** Areas scale by $k^2=4$.

**(d)** Perimeters scale by $k=2$: new perimeter = 24.

$$\boxed{\text{(a) } 7 \text{(b) } \left(-\frac{3}{\sqrt{17}},\frac{12}{\sqrt{17}}\right) \text{(c) } 4 \text{(d) } 24}$$

# Problems 36–50 — Difficult & Real-World

## Problem 36 :

**Answer:**

**(a)** $\parallel u-tv\parallel^2\geq 0$ for any real $t$.
$(a-tc)^2+(b-td)^2\geq 0\Rightarrow(c^2+d^2)t^2-2(ac+bd)t+(a^2+b^2)\geq 0$

**(b)** This quadratic in $t$ has discriminant $\leq 0$:
$\Delta=4(ac+bd)^2-4(c^2+d^2)(a^2+b^2)\leq 0$
$(ac+bd)^2\leq(a^2+b^2)(c^2+d^2)$

**(c)** $(u\cdot v)^2\leq\parallel u\parallel^2\parallel v\parallel^2\Rightarrow\mid u\cdot v\mid\leq\parallel u\parallel\parallel v\parallel$ ✓

**(d)** Equality when $u=tv$ (parallel vectors).
Example: $u=(1,2)$, $v=(2,4)$: $u\cdot v=10$, $\parallel u\parallel\parallel v\parallel=\sqrt{5}\sqrt{20}=10$ ✓

## Problem 37 :

**Answer:**

**(a)** Define $H$ such that $(H-A)\cdot(C-B)=0$ and $(H-B)\cdot(C-A)=0$.

**(b)** Let $H=(x,y)$. Cond 1: $(x,y)\cdot(C-B)=0$.
Cond 2: $(x-4,y)\cdot(C-A)=0$.

**(c)** Add the two equations:
$(H-A)\cdot(C-B)+(H-B)\cdot(C-A)=0$
Expand and rearrange to get $(H-C)\cdot(B-A)=0$ – the third altitude condition.

**(d)** For $A(0,0)$, $B(4,0)$, $C(1,3)$:
$C-B=(-3,3)$, $C-A=(1,3)$.
Cond 1: $-3x+3y=0\Rightarrow y=x$.
Cond 2: $(x-4)+3y=0\Rightarrow x-4+3x=0\Rightarrow 4x=4\Rightarrow x=1$, $y=1$.
$$H=(1,1)$$

## Problem 38 :

**Answer:**

**(a)** $PA^2=(x-1)^2+(y-2)^2$, $PB^2=(x-5)^2+(y-4)^2$, $PC^2=(x-3)^2+(y-8)^2$

**(b)** $(x-1)^2+(y-2)^2+(x-5)^2+(y-4)^2=(x-3)^2+(y-8)^2$
Expand: $x^2-2x+1+y^2-4y+4+x^2-10x+25+y^2-8y+16=x^2-6x+9+y^2-16y+64$
$2x^2-12x+2y^2-12y+46$
$=x^2-6x+y^2-16y+73$
$x^2-6x+y^2+4y-27=0$
$(x-3)^2+(y+2)^2=9+4+27=40$

**(c)** Circle with centre $(3,-2)$, radius $\sqrt{40}$.

**(d)** $G=\left(\frac{1+5+3}{3},\frac{2+4+8}{3}\right)=\left(3,\frac{14}{3}\right)$
Check: $(3-3)^2+\left(\frac{14}{3}+2\right)^2=\left(\frac{20}{3}\right)^2=\frac{400}{9}\neq 40$
$G$ is not on the locus.

$$\boxed{\begin{array}{c}\text{(b) } (x-3)^2+(y+2)^2=40\\ \text{(c) Circle}\\ \text{(d) } G \text{ not on locus}\end{array}}$$

## Problem 39 :

**Answer:**

**(a)** General circle: $x^2+y^2+Dx+Ey+F=0$.
$A(0,0)$: $F=0$.
$B(4,0)$: $16+4D=0\Rightarrow D=-4$.
$C(0,3)$: $9+3E=0\Rightarrow E=-3$.

**(b)** $D=-4$, $E=-3$, $F=0$.

**(c)** $x^2 + y^2 - 4x - 3y = 0 \Rightarrow \left(x{-}2)^2{-}\frac{4}{4}?\right.$
Complete square:
$\left(x - 2)^2 - 4 + (y - \frac{3}{2})^2 - \frac{9}{4} = 0 \Rightarrow \left(x - 2)^2 + (y - \frac{3}{2})^2 = \frac{25}{4}\right.\right.$
Centre: $\left(2, \frac{3}{2}\right)$, radius $\frac{5}{2}$.

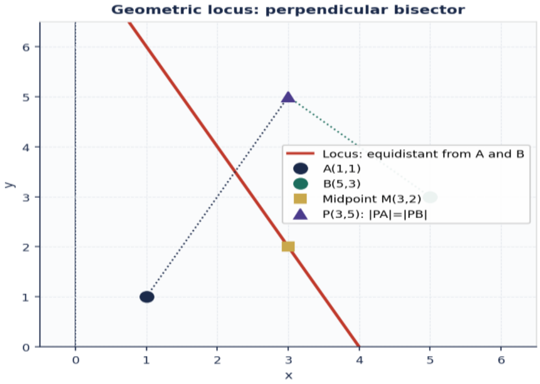
**(d)** Check: $A(0,0)$: $0 = 0$ ✓, $B(4,0)$: $16 - 16 = 0$ ✓, $C(0,3)$: $9 - 9 = 0$ ✓.

$$\boxed{\left(x - 2)^2 + \left(y{-}\frac{3}{2}\right)^2 = \frac{25}{4}\right.}$$

.

## Problem 40 :

Geometric locus: perpendicular bisector



**Answer:**

**(a)** $A(1,0)$, $B(5,4)$. Midpoint $(3, 2)$, slope $AB = 1$, perpendicular slope $-1$.
Line: $y - 2 = -(x - 3) \Rightarrow x + y = 5$. ✓

**(b)** Take $P(2,3)$ on locus: $PA = \sqrt{(1)^2 + (3)^2} = \sqrt{10}$, $PB = \sqrt{9 + 1} = \sqrt{10}$. Equal ✓

**(c)** Substitute $y = 5 - x$ into $x^2 + y^2 = 25$:
$x^2 + (5 - x)^2 = 25 \Rightarrow x^2 + 25 - 10x + x^2 = 25 \Rightarrow 2x^2 - 10x = 0 \Rightarrow 2x(x - 5) = 0$
Points: $(0, 5)$ and $(5, 0)$.

**(d)** These are the points on the perpendicular bisector that also lie on the circle.

$$\boxed{\text{(c) } (0,5), (5,0)}$$

## Problem 41 :

**Answer:**

**(a)** $\overrightarrow{AB} = (4,1)$. $\overrightarrow{DC} = \overrightarrow{AB} \Rightarrow C = D + \overrightarrow{AB} = (2 + 4, 4 + 1) = (6,5)$.

**(b)** Midpoint of $AC$: $\left(\frac{1+6}{2}, \frac{1+5}{2}\right) = \left(\frac{7}{2}, 3\right)$.
Midpoint of $BD$: $\left(\frac{5+2}{2}, \frac{2+4}{2}\right) = \left(\frac{7}{2}, 3\right)$.

**(c)** Equal midpoints → diagonals bisect each other.

**(d)** $\overrightarrow{AB} = (4,1)$, $\overrightarrow{AD} = (1,3)$.
Area $= \mid 4(3) - 1(1) \mid = \mid 12 - 1 \mid = 11$.

$$\boxed{C = (6,5), \text{ area} = 11}$$

## Problem 42 :

**Answer:**

**(a)** $P = (x, 0)$: $PA^2 = x^2 + 9$, $PB^2 = (x - 4)^2 + 1$
$x^2 + 9 = (x - 4)^2 + 1 \Rightarrow x^2 + 9 = x^2 - 8x + 16 + 1 \Rightarrow 9 = -8x + 17 \Rightarrow 8x = 8 \Rightarrow x = 1$
$$P = (1,0)$$

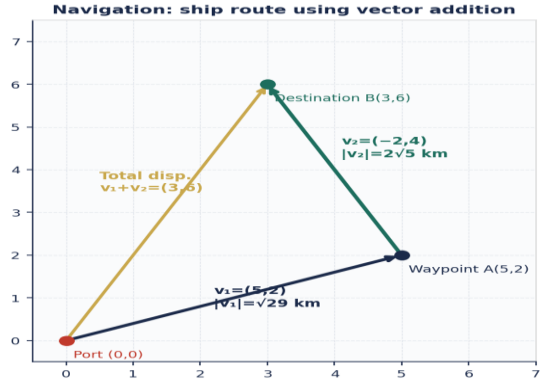
**(b)** $Q = (0, y)$: $QA^2 = (y - 3)^2$, $QB^2 = 16 + (y - 1)^2$
$y^2 - 6y + 9 = 16 + y^2 - 2y + 1 \Rightarrow -6y + 9 = -2y + 17 \Rightarrow -4y = 8 \Rightarrow y = -2$
$$Q = (0, -2)$$

**(c)** Perpendicular bisector of $AB$: midpoint $(2, 2)$, slope $AB = -1/2$, perp slope $= 2$.
Line: $y - 2 = 2(x - 2) \Rightarrow y = 2x - 2$.
Check $P(1,0)$: $0 = 2 - 2$ ✓, $Q(0,-2)$: $-2 = -2$ ✓

$$\boxed{P = (1,0),\ Q = (0,-2),\ y = 2x - 2}$$

## Problem 43 :

Navigation: ship route using vector addition



**Answer:**

**(a)** From graph: $v_1 = (5,2)$, $v_2 = (-2,4)$. Total displacement: $(3, 6)$.

**(b)** $\parallel total \parallel = \sqrt{3^2 + 6^2} = \sqrt{45} = 3\sqrt{5} \approx 6.71$ km.

**(c)** Angle $= \arctan\left(\frac{6}{3}\right) = \arctan(2) \approx 63.4°$ from east.

**(d)** Leg 1: $\parallel v_1 \parallel = \sqrt{29} \approx 5.39$ km, time $= 5.39/15 \approx 0.36$ h.
Leg 2: $\parallel v_2 \parallel = \sqrt{20} \approx 4.47$ km, time $= 4.47/15 \approx 0.30$ h.
Total time $\approx 0.66$ h. Direct: $6.71/15 \approx 0.45$ h.

$$\boxed{\text{(a) } (3,6) \text{(b) } 3\sqrt{5} \text{ km(c) } 63.4° \text{(d) } 0.66 \text{ h vs } 0.45 \text{ h}}$$

## Problem 44 :

**Answer:**

**(a)** $AB = (4,0)$, $\parallel AB \parallel = 4$
$BC = (0,4)$, $\parallel BC \parallel = 4$
$CD = (-4,0)$, $\parallel CD \parallel = 4$
$DA = (0,-4)$, $\parallel DA \parallel = 4$
$AB \cdot BC = 0$ → right angle. Square ✓

**(b)** Diagonal $AC$: from $A(1,2)$ to $C(5,6)$: slope $= 1$, equation $y = x + 1$

Diagonal $BD$: from $B(5,2)$ to $D(1,6)$: slope $= -1$, equation $y = -x + 7$

**(c)** Directions: $(1,1)$ and $(-1,1)$. Dot product $= -1 + 1 = 0 \rightarrow$ perpendicular.
$\| AC \| = \sqrt{4^2 + 4^2} = 4\sqrt{2}$, $\| BD \| = \sqrt{(-4)^2 + 4^2} = 4\sqrt{2} \rightarrow$ equal.

**(d)** Side method: $4^2 = 16$. Diagonal method: $\frac{(4\sqrt{2})^2}{2} = \frac{32}{2} = 16$. ✓

$$\boxed{\text{(b) } y = x + 1,\ y = -x + 7 \text{(d) } 16}$$

## Problem 45 :

**Answer:**

**(a)** $y = 2x$: direction $(1,2)$, unit $u_1 = \frac{(1,2)}{\sqrt{5}}$.
$y = \frac{x}{2}$: direction $(2,1)$, unit $u_2 = \frac{(2,1)}{\sqrt{5}}$.

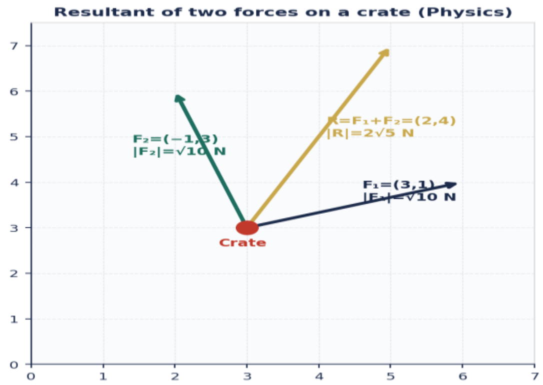
**(b)** $u_1 + u_2 = \frac{(3,3)}{\sqrt{5}} \rightarrow$ direction $(1,1)$.
$u_1 - u_2 = \frac{(-1,1)}{\sqrt{5}} \rightarrow$ direction $(-1,1)$.

**(c)** Bisector 1: $y = x$. Bisector 2: $y = -x$.

**(d)** Directions $(1,1)$ and $(-1,1)$: dot product $= -1 + 1 = 0 \rightarrow$ perpendicular.

$$\boxed{y = x,\ y = -x}$$

## Problem 46 :



**Answer:**

**(a)** From graph: $F_1 = (3,1)$, $F_2 = (-1,3)$. Resultant $R = F_1 + F_2 = (2,4)$.

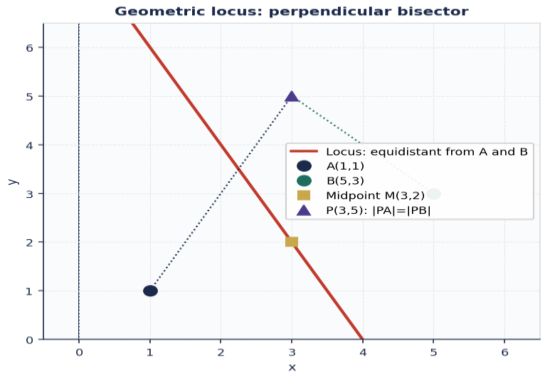
**(b)** $\| R \| = \sqrt{4 + 16} = \sqrt{20} = 2\sqrt{5} \approx 4.47$ N.

**(c)** Angle $= \arctan\left(\frac{4}{2}\right) = \arctan(2) \approx 63.4°$.

**(d)** For equilibrium: $F_3 = -R = (-2,-4)$ N.

$$\boxed{\text{(a) } R = (2,4) \text{(b) } 2\sqrt{5} \text{ N(c) } 63.4° \text{(d) } F_3 = (-2,-4)}$$

## Problem 47 :



**Answer:**

**(a)** $\sqrt{x^2 + (y-2)^2} = y + 2$ (for $y \geq -2$)

**(b)** Square: $x^2 + (y-2)^2 = (y+2)^2 \Rightarrow x^2 + y^2 - 4y + 4 = y^2 + 4y + 4 \Rightarrow x^2 = 8y$

**(c)** $y = \frac{x^2}{8}$. Vertex $(0,0)$, axis $x = 0$ (y-axis), opens upward.

**(d)** $x^2 = 8(4) = 32 \Rightarrow x = \pm 4\sqrt{2}$
Points: $\left(4\sqrt{2}, 4\right)$ and $\left(-4\sqrt{2}, 4\right)$

$$\boxed{y = \frac{x^2}{8}, \text{ points } (\pm 4\sqrt{2}, 4)}$$

## Problem 48 :

**Answer:**

**(a)** $A(0,4)$, $B(-3,0)$, $C(3,0)$.
$AB = \sqrt{9 + 16} = 5$, $AC = \sqrt{9 + 16} = 5$. Isosceles!

**(b)** $\frac{BD}{DC} = \frac{5}{5} = 1 \Rightarrow D$ is the midpoint of $BC = (0,0)$.

**(c)** Angle bisector from $A(0,4)$ to $D(0,0)$: vertical line $x = 0$.

**(d)** Line $AB$: through $A(0,4)$ and $B(-3,0)$: slope $= \frac{4}{3}$, equation $4x - 3y + 12 = 0$.
Distance from $D(0,0)$: $\frac{|12|}{\sqrt{16+9}} = \frac{12}{5}$.
Line $AC$: through $A(0,4)$ and $C(3,0)$: slope $= -\frac{4}{3}$, equation $4x + 3y - 12 = 0$.
Distance from $D$: $\frac{|-12|}{5} = \frac{12}{5}$. Equal ✓

## Problem 49 :

**Answer:**

**(a)** Let $A(0,0)$, $B(d,0)$. $PA^2 = k^2 PB^2$:
$$x^2 + y^2 = k^2[(x-d)^2 + y^2]$$

**(b)** Expand: $x^2 + y^2 = k^2(x^2 - 2dx + d^2 + y^2)$
$$(1-k^2)(x^2 + y^2) + 2k^2 dx - k^2 d^2 = 0$$

**(c)** Divide by $(1-k^2)$: $x^2 + y^2 + \frac{2k^2 d}{1-k^2}x - \frac{k^2 d^2}{1-k^2} = 0$

Complete square: $\left(x + \frac{k^2 d}{1-k^2}\right)^2 + y^2 = \frac{k^2 d^2}{(1-k^2)^2}$

Centre: $\left(-\frac{k^2 d}{1-k^2}, 0\right) = \left(\frac{k^2 d}{k^2-1}, 0\right)$, radius $= \frac{kd}{|1-k^2|}$.

**(d)** For $k = 2$, $d = 3$: centre $\left(\frac{4(3)}{3}, 0\right) = (4,0)$,

radius $\sqrt{\frac{4(9)}{9}} = 2$.
Circle: $(x-4)^2 + y^2 = 4$.

$$\boxed{(x-4)^2 + y^2 = 4}$$

## Problem 50 :

**Answer:**

**(a)** $AC$: length $\sqrt{144+81} = 15$ m. Slope $= \frac{9}{12} = \frac{3}{4}$.
Equation: $y = \frac{3}{4}x$.

**(b)** $BD$: length $15$ m. Slope $= \frac{9-0}{0-12} = -\frac{3}{4}$.
Equation: $y = -\frac{3}{4}x + 9$.

**(c)** Intersection: $\frac{3x}{4} = -\frac{3x}{4} + 9 \Rightarrow \frac{3x}{2} = 9 \Rightarrow x = 6$, $y = 4.5$. Centre: $(6, 4.5)$.

**(d)** $E = (6,0)$, $F = (6,9)$. $EF$ is vertical $x = 6$. $AD$ is vertical $x = 0 \Rightarrow EF \parallel AD$.

**(e)** $G = (12,4.5)$, $H = (0,4.5)$. $GH$ is horizontal $y = 4.5$. $AB$ is horizontal $y = 0 \Rightarrow GH \parallel AB$.
$EF \cap GH$: $(6, 4.5)$ — the centre.

$$\boxed{\text{(a) } y = \frac{3x}{4} \text{ (b) } y = -\frac{3x}{4} + 9 \text{ (c) } (6,4.5) \text{ (d) } x = 6 \text{ (e) } (6,4.5)}$$

### CHAPTER 8 — PLANE GEOMETRY & TRANSFORMATIONS

# SOLUTION OF PROBLEMS

## Problems 1–10 — Instant Recall

### Problem 1 :

**Answer:**

**(a)** $c = \sqrt{a^2 + b^2} = \sqrt{5^2 + 12^2} = \sqrt{25 + 144} = \sqrt{169} = 13$

**(b)** $b = \sqrt{c^2 - a^2} = \sqrt{17^2 - 8^2} = \sqrt{289 - 64} = \sqrt{225} = 15$

**(c)** $c = \sqrt{7^2 + 7^2} = \sqrt{49 + 49} = \sqrt{98} = 7\sqrt{2}$

**(d)** $b = \sqrt{c^2 - a^2} = \sqrt{2^2 - 1^2} = \sqrt{4 - 1} = \sqrt{3}$

$$\boxed{\text{(a) } 13 \text{(b) } 15 \text{(c) } 7\sqrt{2} \text{(d) } \sqrt{3}}$$

### Problem 2 :

**Answer:**

**(a)** $8^2 + 15^2 = 64 + 225 = 289 = 17^2$. **Yes, right-angled.**

**(b)** $5^2 + 7^2 = 25 + 49 = 74 \neq 81 = 9^2$. **Not right-angled.**

**(c)** $9^2 + 40^2 = 81 + 1600 = 1681 = 41^2$. **Yes, right-angled.**

**(d)** $6^2 + 8^2 = 36 + 64 = 100 \neq 121 = 11^2$. **Not right-angled.**

$$\boxed{\text{(a) Yes(b) No(c) Yes(d) No}}$$

### Problem 3 :

**Answer:**

**(a)** $k = \frac{A'B'}{AB} = \frac{9}{6} = \frac{3}{2}$

**(b)** $B'C' = k \cdot BC = \frac{3}{2} \cdot 4 = 6$

$$C'A' = k \cdot CA = \frac{3}{2} \cdot 5 = \frac{15}{2} = 7.5$$

**(c)** $P' = k \cdot P = \frac{3}{2} \cdot 15 = \frac{45}{2} = 22.5$

$$\boxed{\text{(a) } k = \frac{3}{2} \text{(b) } B'C' = 6,\ C'A' = 7.5 \text{(c) } 22.5}$$

### Problem 4 :

**Answer:**

By Thales' theorem: $\frac{AD}{DB} = \frac{AE}{EC}$ and $\frac{AD}{AB} = \frac{DE}{BC}$.

**(a)** $\frac{4}{6} = \frac{5}{EC} \Rightarrow 4EC = 30 \Rightarrow EC = 7.5$

**(b)** $\frac{AD}{AB} = \frac{DE}{BC} \Rightarrow \frac{3}{9} = \frac{4}{BC} \Rightarrow 3BC = 36 \Rightarrow BC = 12$

**(c)** $AD = AB - DB = 10 - 4 = 6$
$\frac{AD}{DB} = \frac{AE}{EC} \Rightarrow \frac{6}{4} = \frac{6}{EC} \Rightarrow 6EC = 24 \Rightarrow EC = 4$

$$\boxed{\text{(a) } 7.5 \text{(b) } 12 \text{(c) } 4}$$

### Problem 5 :

**Answer:**

**(a)** Reflection across the $x$-axis (the $y$-coordinates change sign).

**(b)** Translation by vector $(3, 0)$ (shift right by 3 units).

**(c)** Homothety (scaling) with centre $O(0,0)$ and ratio $k = 2$ (all coordinates doubled).

**(d)** Reflection across the line $y = x$ (the coordinates are swapped).

### Problem 6 :

**Answer:**

**(a)** Reflection across x-axis: $(x, y) \to (x, -y)$.
$$P'(3,1)$$

**(b)** Translation by $(-3,4)$: $(x, y) \to (x - 3, y + 4)$.
$$P'(-1,9)$$

**(c)** Rotation $90^\circ$ CCW: $(x, y) \to (-y, x)$.
$$P'(-2,4)$$

**(d)** Homothety with $k = -2$: $(x, y) \to (-2x, -2y)$.
$$P'(-4, -6)$$

### Problem 7 :

**Answer:**

**(a)** Reflection across y-axis: $(x, y) \to (-x, y)$.
$A'(-1,0)$, $B'(-3,0)$, $C'(-2,3)$

**(b)** Rotation by $180^\circ$: $(x, y) \to (-x, -y)$.
$A'(-1,0)$, $B'(-3,0)$, $C'(-2,-3)$

**(c)** Translation by $(2, -1)$: $(x, y) \to (x + 2, y - 1)$.
$A'(3, -1)$, $B'(5, -1)$, $C'(4,2)$

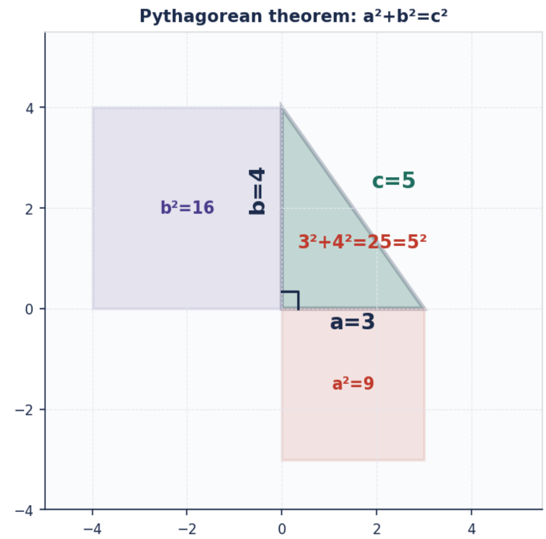
$$\boxed{\begin{array}{c}\text{(a) } (-1,0), (-3,0), (-2,3) \\ \text{(b) } (-1,0), (-3,0), (-2,-3) \\ \text{(c) } (3,-1), (5,-1), (4,2)\end{array}}$$

### Problem 8 :

**Answer:** **(a)** From the figure: $a = 3$, $b = 4$, $c = 5$.
$3^2 + 4^2 = 9 + 16 = 25 = 5^2$ ✓

**(b)** $c = \sqrt{9^2 + 12^2} = \sqrt{81 + 144} = \sqrt{225} = 15$ cm

**(c) $7^2 + 24^2 = 49 + 576 = 625 = 25^2$. Yes, the triangle is right-angled with hypotenuse $25$.**



## Problem 9 :

**Answer:**

**(a)** Perimeter scales by $k$.
$P_{\text{larger}} = 20 \cdot \frac{3}{2} = 30$ cm

**(b)** Area scales by $k^2 = \frac{9}{4}$.
$A_{\text{larger}} = 16 \cdot \frac{9}{4} = 36$ cm²

**(c)** $A_{\text{smaller}} = 81 \cdot \frac{4}{9} = 36$ cm²

$$\boxed{\text{(a) } 30\text{ cm} \text{(b) } 36\text{ cm}^2 \text{(c) } 36\text{ cm}^2}$$

## Problem 10 :

**Answer:**

**(a)** Perpendicular bisector of $AB$.
Midpoint of $AB$ is $(2,0)$. The line is vertical $x = 2$.
Equation: $x = 2$

**(b)** Circle with centre $C(1,2)$ and radius $3$.
Equation: $(x-1)^2 + (y-2)^2 = 9$

**(c)** By Thales' theorem, points $P$ lie on a circle with diameter $AB$.
Centre is $(0,0)$, radius $2$.
Equation: $x^2 + y^2 = 4$

$$\boxed{\text{(a) } x = 2 \text{(b) } (x-1)^2 + (y-2)^2 = 9 \text{(c) } x^2 + y^2 = 4}$$

# Problems 11–35 — Medium

## Problem 11 :

**Answer:**

**(a)** From the graph: $A(1,5)$, $B(0,0)$, $C(6,0)$.
$D$ and $E$ are points on $AB$ and $AC$ respectively such that $DE \parallel BC$.
The ratio is approximately $0.55$:

$$\begin{aligned} D &= A + 0.55(B - A) = (1 - 0.55, 5 - 2.75) \\ &= (0.45, 2.25) \\ E &= A + 0.55(C - A) = (1 + 2.75, 5 - 2.75) \\ &= (3.75, 2.25) \end{aligned}$$

**(b)** $AD = 0.55 \cdot AB$, $DB = 0.45 \cdot AB$.
$AE = 0.55 \cdot AC$, $EC = 0.45 \cdot AC$.
$\frac{AD}{DB} = \frac{0.55}{0.45} = \frac{11}{9}$, $\frac{AE}{EC} = \frac{11}{9}$. Equal ✓

**(c)** $\frac{DE}{BC} = \frac{AD}{AB} = 0.55$ ✓

**(d)** Area ratio $= k^2 = (0.55)^2 = 0.3025$.

$$\frac{\text{Area}(ADE)}{\text{Area}(ABC)} \approx 0.30$$

## Problem 12 :

**Answer:**

**(a)** Assume $DE$ is not parallel to $BC$. Through $D$, draw $DE' \parallel BC$ meeting $AC$ at $E'$.

**(b)** By Thales' theorem: $\frac{AD}{DB} = \frac{AE'}{E'C}$.

**(c)** But we are given $\frac{AD}{DB} = \frac{AE}{EC}$. So $\frac{AE}{EC} = \frac{AE'}{E'C}$.

**(d)** This implies $\frac{AE}{AC-AE} = \frac{AE'}{AC-AE'}$. Solving gives $AE = AE'$.
So $E = E'$. Hence $DE$ passes through $E' = E$ and is parallel to $BC$. ✓

## Problem 13 :

**Answer:**

**(a)** Right triangle with hypotenuse $5$ m, base $3$ m, height $h$.

**(b)** $h = \sqrt{5^2 - 3^2} = \sqrt{25 - 9} = \sqrt{16} = 4$ m

**(c)** New base $= 4$ m. New height $= \sqrt{5^2 - 4^2} = \sqrt{25 - 16} = \sqrt{9} = 3$ m.
Drop $= 4 - 3 = 1$ m.

**(d)** In this case, the drop equals the increase (1 m). But this is not general.
For example, from base $0$ to $1$: drop $= \sqrt{25 - 0} - \sqrt{25 - 1} \approx 5 - 4.899 \approx 0.101$ m, which is much smaller than the increase of 1 m. The relationship is non-linear.

$$\boxed{\text{(b) } 4\text{ m} \text{(c) } 1\text{ m} \text{(d) Not generally equal}}$$

## Problem 14 :

**Answer:**

**(a)** The rod and the tree form two right triangles with the ground. The sun's rays are parallel, so the triangles are similar.

**(b)** Similarity ratio: $\frac{H}{8} = \frac{2}{3.2}$
$H = \frac{2 \cdot 8}{3.2} = \frac{16}{3.2} = 5$ m

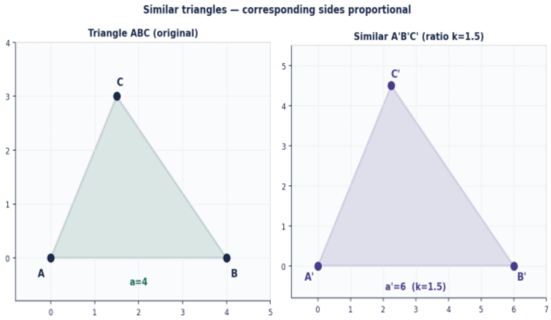
**(c)** $H = 8 \cdot \tan(32^\circ) \approx 8 \cdot 0.625 = 5$ m ✓

**(d)** Tree height is unchanged: $H = 5$ m.
Shadow $= \frac{H}{\tan(45^\circ)} = \frac{5}{1} = 5$ m

$$\boxed{\text{(b) 5 m(d) 5 m}}$$

.

## Problem 15 :



**Answer:**

**(a)** $AB = 4$, $BC = \sqrt{(4-1.5)^2 + (0-3)^2} = \sqrt{6.25+9} = \sqrt{15.25} \approx 3.91$,

$$CA = \sqrt{(1.5)^2 + 9} \approx 3.35$$

**(b)** $A'B' = 6$, $B'C' \approx 5.87$, $C'A' \approx 5.02$

**(c)** $\cos(A) = \frac{AB^2+CA^2-BC^2}{2\cdot AB\cdot CA} = \frac{16+11.23-15.25}{2\cdot 4\cdot 3.35} \approx \frac{11.98}{26.8} \approx 0.447$
Same for the larger triangle → angles equal.

**(d)** Area$(ABC) = \frac{1}{2}\cdot 4\cdot 3 = 6$

$$\text{Area}(A'B'C') = 6\cdot k^2 = 6\cdot\frac{9}{4} = 13.5$$

Ratio $= \frac{13.5}{6} = \frac{9}{4} = k^2$ ✓

.

## Problem 16 :

**Answer:**

**(a)** Let $M_a$ be the midpoint of $BC$, $M_b$ the midpoint of $CA$, $M_c$ the midpoint of $AB$.
By the midpoint theorem, $M_bM_c \parallel BC$ and $M_bM_c = \frac{1}{2}BC$.

**(b)** Similarly, $M_cM_a = \frac{1}{2}CA$ and $M_aM_b = \frac{1}{2}AB$.
So each side of the midpoint triangle is parallel to and half the length of a corresponding side of $ABC$.
Therefore, the midpoint triangle is similar to $ABC$ with $k = \frac{1}{2}$.

**(c)** Area ratio $= k^2 = \frac{1}{4}$.

**(d)** If Area$(ABC) = 24$ cm²,
then Area$(M_aM_bM_c) = 24\cdot\frac{1}{4} = 6$ cm².

$$\boxed{\text{(a)-(b) Similar with } k = \frac{1}{2}\text{(c) }\frac{1}{4}\text{(d) 6 cm}^2}$$

## Problem 17 :

**Answer:**

**(a)** In right triangle $ABC$ (right angle at $C$), altitude $CD$ meets hypotenuse $AB$ at $D$.
Triangles $ACD$ and $CBD$ are each similar to $ABC$ (they share an acute angle).

**(b)** From the similarity of $ACD$ and $CBD$:
$\frac{AD}{CD} = \frac{CD}{DB} \Rightarrow \frac{p}{h} = \frac{h}{q} \Rightarrow h^2 = p\cdot q$ ✓

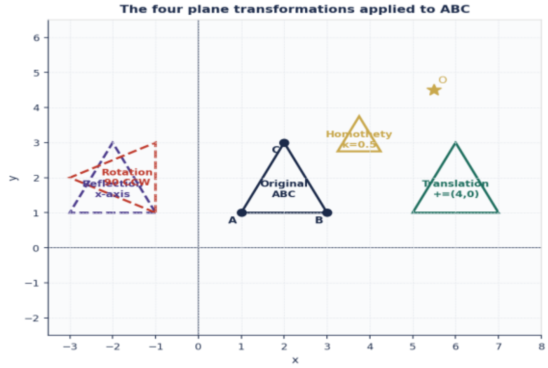
**(c)** Also:
From $CBD \sim ABC$: $\frac{a}{c} = \frac{q}{a} \Rightarrow a^2 = cq$
From $ACD \sim ABC$: $\frac{b}{c} = \frac{p}{b} \Rightarrow b^2 = cp$

**(d)** For $c = 20$, $p = 4$, $q = 16$:

$$h = \sqrt{4\cdot 16} = \sqrt{64} = 8$$
$$a = \sqrt{20\cdot 16} = \sqrt{320} = 8\sqrt{5}$$
$$b = \sqrt{20\cdot 4} = \sqrt{80} = 4\sqrt{5}$$

$$\boxed{h = 8,\ a = 8\sqrt{5},\ b = 4\sqrt{5}}$$

## Problem 18 :



**Answer:**

**(a)** Identify from the graph:
$T_1$: Translation, $T_2$: Reflection, $T_3$: Rotation, $T_4$: Homothety.

**(b)** Distance preserved: translation, reflection, rotation (isometries).
Orientation preserved: translation and rotation (direct isometries).
Reflection reverses orientation. Homothety preserves angles but not distances.

**(c)** Translation/reflection/rotation: congruent.
Homothety: similar (since $k = 0.5$).

**(d)** Area scales by $k^2 = 0.25$.

$$\text{Area}(T_4) = 4\cdot 0.25 = 1$$

## Problem 19 :

**Answer:**

**(a)** Reflection across $y = 2$: $(x, y) \to (x, 4-y)$.
$P'(3,0)$. $A'(1,2)$, $B'(5,2)$ (on mirror line, unchanged).

**(b)** Rotation by $180^\circ$ about $M(3,2)$: $(x, y) \to (6-x, 4-y)$.
$P'(3,0)$. $A'(5,2)$, $B'(1,2)$ (AB maps to itself reversed).

**(c)** Homothety centre $M(3,2)$, $k = 2$: $X' = M + 2(X - M)$.

$$P' = (3,2) + 2(0,2) = (3,6)$$
$$A' = (3,2) + 2(-2,0) = (-1,2)$$
$$B' = (3,2) + 2(2,0) = (7,2)$$

$$\boxed{\begin{array}{c}\text{(a) } P'(3,0),\ A'(1,2),\ B'(5,2)\\ \text{(b) } P'(3,0),\ A'(5,2),\ B'(1,2)\\ \text{(c) } P'(3,6),\ A'(-1,2),\ B'(7,2)\end{array}}$$

## Problem 20 :

**Answer:**

**(a)** Let the parallel lines be $x = 0$ and $x = d$.
Reflection across $x = 0$: $P(x,y) \rightarrow P_1(-x,y)$.
Reflection across $x = d$: $P_1(-x,y) \rightarrow P_2(2d - (-x),y) = (2d + x,y)$.

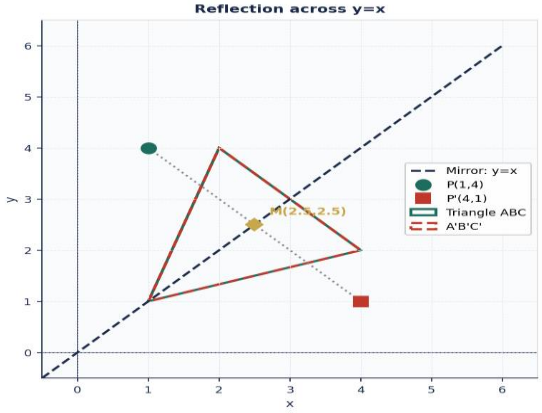
**(b)** The vector from $P$ to $P_2$ is $(2d,0)$.
This is independent of the position of $P$.

**(c)** Therefore, every point is translated by $(2d,0)$ → composition is a translation.

**(d)** For lines $y = 1$ and $y = 5$, distance $d = 4$.
Translation by $(0,2d) = (0,8)$.

$$\boxed{\text{Translation by } (2d,0)}$$

## Problem 21 :



**Answer:**

**(a)** $P(1,4)$, $P'(4,1)$. Reflection across $y = x$: $(x,y) \rightarrow (y,x)$. ✓

**(b)** Each vertex $(x,y)$ maps to $(y,x)$. The coordinates are swapped.

**(c)** Midpoint of $PP'$: $\left(\frac{1+4}{2},\frac{4+1}{2}\right) = \left(\frac{5}{2},\frac{5}{2}\right)$.
This point lies on $y = x$ ✓

**(d)** $PP'$ direction: $(3,-3)$, slope $-1$.
Mirror line $y = x$ has slope 1. Product $= -1$ → perpendicular ✓

## Problem 22 :

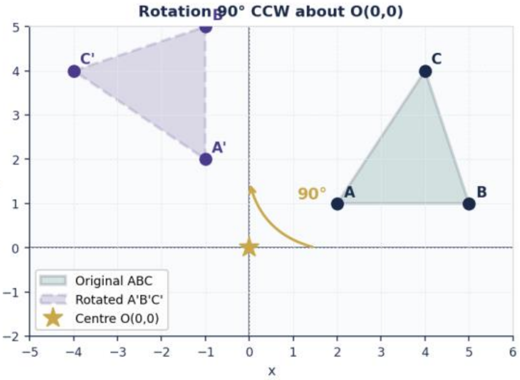
**Answer:**

**(a)** $90^\circ + 90^\circ = 180^\circ$.
$R_2 \circ R_1$ is rotation by $180^\circ$: $(x,y) \rightarrow (-x,-y)$.

**(b)** $R_1(3,1) = (-1,3)$.
$R_2(-1,3) = (-3,-1)$.
Direct: $R_{180}(3,1) = (-3,-1)$. ✓

**(c)** $R_4 \circ R_3$: $120^\circ + 120^\circ = 240^\circ$ rotation.
$R_3 \circ R_4 \circ R_3$: three $120^\circ$ rotations = $360^\circ$ = identity.

## Problem 23 :



**Answer:**

**(a)** $A(2,1) \rightarrow A'(-1,2)$, $B(5,1) \rightarrow B'(-1,5)$, $C(4,4) \rightarrow C'(-4,4)$.
Rule: $(x,y) \rightarrow (-y,x)$. ✓

**(b)** $\parallel OA \parallel = \sqrt{4+1} = \sqrt{5}$, $\parallel OA' \parallel = \sqrt{1+4} = \sqrt{5}$. Equal ✓

**(c)** $OA \cdot OA' = (2,1) \cdot (-1,2) = -2 + 2 = 0 \rightarrow$ perpendicular → $90^\circ$ ✓

**(d)** Direct isometry: orientation is preserved (clockwise order remains clockwise).

## Problem 24 :

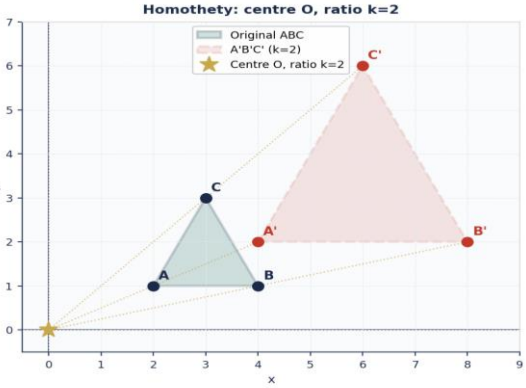
**Answer:**

**(a)** Let $R$ be a rotation by angle $2\alpha$ about $O$.
$R = S_2 \circ S_1$ where $S_1$ and $S_2$ are reflections across two lines through $O$ making angle $\alpha$.

**(b)** Verify for $60^\circ$: $S_1$ across x-axis, $S_2$ across line at $30^\circ$.

**(c)** $S_1(1,0) = (1,0)$.
$S_2$ across line at $30^\circ$: formula $S_2(x,y) = (x\cos 60^\circ + y\sin 60^\circ, x\sin 60^\circ - y\cos 60^\circ)$.
$S_2(1,0) = \left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$.
Rotation by $60^\circ$: $R_{60}(1,0) = (\cos 60^\circ, \sin 60^\circ) = \left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$. ✓

## Problem 25 :



**Answer:**

**(a)** $A(2,1)$, $A'(4,2)$. $OA' = 2 \cdot OA$ ✓

**(b)** $A$, $O$, $A'$ are collinear since $OA'$ is a scalar multiple of $OA$.

**(c)** Area scales by $k^2 = 4$.
Area$(A'B'C') = 4 \cdot$ Area$(ABC)$.

**(d)** $G = \frac{A+B+C}{3} = \left(3, \frac{5}{3}\right)$.
$G' = k \cdot G = 2\left(3, \frac{5}{3}\right) = \left(6, \frac{10}{3}\right)$. ✓

## Problem 26 :

**Answer:**

**(a)** Draw a line through $C$ parallel to $AD$, meeting the extension of $BA$ at $E$.
$\angle ECA = \angle DAC$ (alternate interior angles).
$\angle DAB = \angle AEC$ (corresponding angles).
Since $AD$ bisects angle $A$, $\angle DAB = \angle DAC$, so $\angle ECA = \angle AEC$.

**(b)** Triangle $AEC$ is isosceles with $AE = AC$.

**(c)** By Thales' theorem (since $CE \parallel AD$):
$\frac{BD}{DC} = \frac{BA}{AE} = \frac{BA}{AC}$. ✓

## Problem 27 :

**Answer:**

**(a)** $d = 5$

**(b)** $r_1 + r_2 = 5 = d \rightarrow$ externally tangent.

**(c)** The point of tangency lies on the line joining the centres, at distance $r_1 = 3$ from $O_1$: $T(3,0)$.

**(d)** $\mid r_1 - r_2 \mid < d < r_1 + r_2$

$$\boxed{\begin{array}{c}\text{(a) } 5\\ \text{(b) Externally tangent}\\ \text{(c) } (3,0)\\ \text{(d) } \mid r_1 - r_2 \mid < d < r_1 + r_2\end{array}}$$

## Problem 28 :

**Answer:**

**(a)** A regular hexagon is made of 6 equilateral triangles, each with side length $r$.
So each side of the hexagon is $r$.

**(b)** Area of one equilateral triangle $= \frac{\sqrt{3}}{4} r^2$.

Total area $= 6 \cdot \frac{\sqrt{3}}{4} r^2 = \frac{3\sqrt{3}}{2} r^2$.

**(c)** Apothem $= r\cos(30^\circ) = \frac{\sqrt{3}}{2} r$.

**(d)** $\frac{\text{Area(hexagon)}}{\text{Area(circle)}} = \frac{\frac{3\sqrt{3}}{2} r^2}{\pi r^2} = \frac{3\sqrt{3}}{2\pi} \approx 0.827$

$$\boxed{\text{(a) } r \text{(b) } \frac{3\sqrt{3}}{2} r^2 \text{(c) } \frac{\sqrt{3}}{2} r \text{(d) } \frac{3\sqrt{3}}{2\pi}}$$

## Problem 29 :

**Answer:**

**(a)** Consider the homothety with centre $G$ (centroid) and ratio $k = -\frac{1}{2}$.
This maps each vertex to the midpoint of the opposite side (the medial triangle).

**(b)** The circumcentre $O$ of the medial triangle maps to a point on each altitude.

**(c)** This image point is the same for all three altitudes: the orthocentre $H$.
Thus the altitudes are concurrent.

**(d)** For $A(0,0)$, $B(4,0)$, $C(1,3)$:
$G = \left(\frac{5}{3}, 1\right)$.
Altitude from $A$: slope of $BC = -1$, altitude slope 1, equation $y = x$.
Altitude from $B$: slope of $AC = 3$, altitude slope $-\frac{1}{3}$, equation $y = -\frac{1}{3}(x - 4)$.
Intersection: $x = -\frac{x}{3} + \frac{4}{3} \Rightarrow \frac{4x}{3} = \frac{4}{3} \Rightarrow x = 1$, $y = 1$.
$$H = (1,1)$$

## Problem 30 :

**Answer:**

**(a)** Let $G_1$ be the point dividing median from $A$ in ratio $2:1$:
$G_1 = A + \frac{2}{3}(M_a - A)$, where $M_a = \frac{B+C}{2}$.
$G_1 = A + \frac{2}{3}\left(\frac{B+C}{2} - A\right) = A + \frac{B+C}{3} - \frac{2A}{3} = \frac{A+B+C}{3}$.

**(b)** Similarly, the point dividing the median from $B$ in ratio $2:1$ is also $\frac{A+B+C}{3}$.

**(c)** So all three medians pass through $G = \frac{A+B+C}{3}$.

**(d)** For $A(0,0)$, $B(6,0)$, $C(0,4)$:
$G = \left(\frac{6}{3}, \frac{4}{3}\right) = (2, \frac{4}{3})$.
Median from $A$ to $M_a = (3,2)$: parametric $(0,0) + t(3,2)$. At $t = \frac{2}{3}$: $(2, \frac{4}{3}) = G$ ✓

## Problem 31 :

**Answer:**

**(a)** 3 axes: one through each vertex and the midpoint of the opposite side.

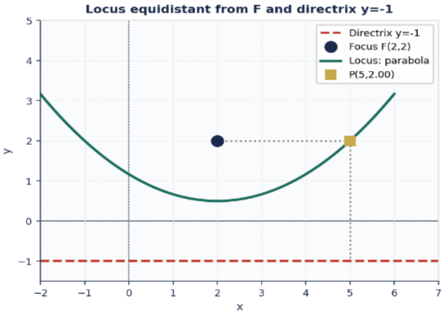
**(b)** 4 axes: 2 through opposite midpoints, 2 through opposite vertices.

**(c)** 6 axes: 3 through opposite vertices, 3 through midpoints of opposite sides.

**(d)** 2 axes: horizontal and vertical centre lines.

$$\boxed{\text{(a) } 3 \text{(b) } 4 \text{(c) } 6 \text{(d) } 2}$$

## Problem 32 :



**Answer:**

**(a)** $\sqrt{(x-2)^2+(y-2)^2} = y+1$ (distance to directrix $y=-1$).

**(b)** Square: $(x-2)^2+(y-2)^2=(y+1)^2$

$$(x-2)^2+y^2-4y+4=y^2+2y+1$$

$(x-2)^2=6y-3 \Rightarrow y=\frac{(x-2)^2+3}{6}$

**(c)** Vertex when $x=2$: $y=\frac{3}{6}=\frac{1}{2}$. Vertex: $\left(2,\frac{1}{2}\right)$.

**(d)** $y=5$: $\frac{(x-2)^2+3}{6}=5 \Rightarrow (x-2)^2=27 \Rightarrow x=2\pm 3\sqrt{3}$

$$\boxed{\text{(b) } y=\frac{(x-2)^2+3}{6} \text{ (c) } (2,\frac{1}{2}) \text{(d) } (2\pm 3\sqrt{3},5)}$$

## Problem 33 :

**Answer:**

**(a)** $T(P)=(1+3,4+2)=(4,6)$.
$S(4,6)=(4,-6)$.

$$S\circ T(P)=(4,-6)$$

**(b)** $S(P)=(1,-4)$.
$T(1,-4)=(4,-2)$.

$$T\circ S(P)=(4,-2)$$

**(c)** Different: $(4,-6)\neq(4,-2)$. $T$ and $S$ do not commute.

**(d)** Fixed point of $S\circ T$: $(x,y)\to(x+3,-(y+2))$.
For fixed: $x+3=x \Rightarrow$ impossible. No fixed point.

## Problem 34 :

**Answer:**

**(a)** Homothety with $k=-1$: $X'=G-(X-G)=2G-X$.

$$A'=2G-A=(6,\frac{8}{3})-(0,0)=(6,\frac{8}{3})$$
$$B'=2G-B=(6,\frac{8}{3})-(6,0)=(0,\frac{8}{3})$$
$$C'=2G-C=(6,\frac{8}{3})-(3,4)=(3,-\frac{4}{3})$$

**(b)** A homothety with ratio $k=-1$ is a central symmetry (rotation by $180°$).

**(c)** Since $\mid k \mid = 1$, side lengths are unchanged → congruent.

**(d)** Centroid of $A'B'C'=\frac{A'+B'+C'}{3}=\left(\frac{6+0+3}{3},\frac{\frac{8}{3}+\frac{8}{3}-\frac{4}{3}}{3}\right)=\left(3,\frac{4}{3}\right)=G$ ✓

.

## Problem 35 :

**Answer:**

**(a)** Triangles $DBE$ and $ECD$ share base $DE$ and have equal heights (since $BC \parallel DE$).
So Area$(DBE)$ = Area$(ECD)$.

**(b)** $\frac{\text{Area}(ADE)}{\text{Area}(DBE)}=\frac{AD}{DB}$ (same height from $E$).

**(c)** $\frac{\text{Area}(ADE)}{\text{Area}(ECD)}=\frac{AE}{EC}$ (same height from $D$).

**(d)** Since Area$(DBE)$ = Area$(ECD)$, we get $\frac{AD}{DB}=\frac{AE}{EC}$.
Then $\frac{AD}{AB}=\frac{AE}{AC}$ follows by adding 1 to both sides. ✓

# Problems 36–50 — Difficult & Real-World

## Problem 36 :

**Answer:**

**(a)** In right triangle $ABC$ (right angle at $C$), altitude $CD$ meets hypotenuse $AB$ at $D$.
Triangles $ACD$ and $CBD$ are similar to $ABC$.

**(b)** From $ABC\sim CBD$: $\frac{BC}{AB}=\frac{BD}{BC}\Rightarrow a^2=c\cdot BD$.
From $ABC\sim ACD$: $\frac{AC}{AB}=\frac{AD}{AC}\Rightarrow b^2=c\cdot AD$.

**(c)** Add: $a^2+b^2=c(BD+AD)=c\cdot AB=c^2$. ✓

**(d)** For $a=3, b=4, c=5$:
$BD=\frac{a^2}{c}=\frac{9}{5}$, $AD=\frac{b^2}{c}=\frac{16}{5}$.
$BD+AD=\frac{25}{5}=5=c$.
$9+16=25$ ✓

## Problem 37 :

**Answer:**

**(a)** Place the right triangle with $C(0,0)$, $A(2a,0)$, $B(0,2b)$.
Midpoint $M$ of $AB$ is $(a,b)$.

**(b)** $CM=\sqrt{a^2+b^2}$

**(c)** $AB=\sqrt{(2a)^2+(2b)^2}=2\sqrt{a^2+b^2}$.
$\frac{AB}{2}=\sqrt{a^2+b^2}=CM$. ✓

**(d)** This means $C$ lies on the circle with diameter $AB$ (Thales' theorem).

## Problem 38 :

**Answer:**

**(a)** A rhombus is a parallelogram, so its diagonals bisect each other.

**(b)** Let $O$ be the intersection of diagonals.
In triangles $AOB$ and $COB$:
$\| AO \| = \| CO \|$ (diagonal bisected).
$\| AB \| = \| CB \|$ (all sides equal in a rhombus).
$\| OB \| = \| OB \|$ (shared).

**(c)** By SSS, triangles $AOB \cong COB$.
So $\angle AOB=\angle COB$. Since they are supplementary, each is $90°$.
Thus diagonals are perpendicular.

**(d)** For rhombus with $A(0,2)$, $B(3,0)$, $C(0,-2)$, $D(-3,0)$:
Diagonal $AC$ is vertical (slope undefined).

Diagonal $BD$ is horizontal (slope 0).
They are perpendicular. ✓

## Problem 39 :

**Answer:**

**(a)** For $DE \parallel BC$, we need $\frac{AD}{DB} = \frac{AE}{EC}$.

**(b)** $\frac{x}{8} = \frac{x+2}{12}$

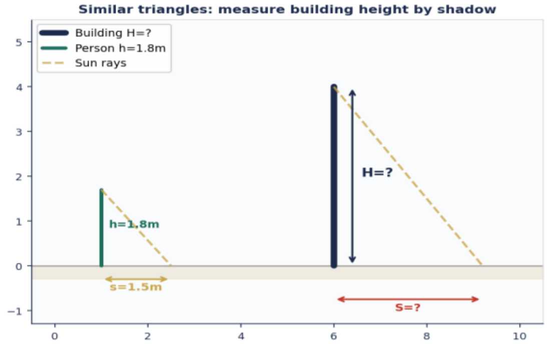
**(c)** $12x = 8(x+2) \Rightarrow 12x = 8x + 16 \Rightarrow 4x = 16 \Rightarrow x = 4$

**(d)** Check: $\frac{4}{8} = \frac{1}{2}, \frac{6}{12} = \frac{1}{2}$. ✓

$$\boxed{x = 4}$$

.

## Problem 40 :



**Answer:**

**(a)** Two similar right triangles: person + shadow, building + shadow.

**(b)** $\frac{H}{12} = \frac{1.8}{1.5} \Rightarrow H = \frac{1.8\cdot 12}{1.5} = 14.4$ m

**(c)** New ratio: $\frac{s'}{s} = \frac{2.4}{1.5} = 1.6$.
$S' = 12 \cdot 1.6 = 19.2$ m

**(d)** $H = \frac{h\cdot S}{s}$. Dimensions: $\left[m\right] \cdot \frac{[m]}{[m]} = \left[m\right]$. Correct.

$$\boxed{\text{(b) } 14.4 \text{ m(c) } 19.2 \text{ m(d) } H = h \cdot \frac{S}{s}}$$

## Problem 41 :

**Answer:**

**(a)** $y = \frac{x^2}{8} \Rightarrow x^2 = 8y$.
$4p = 8 \Rightarrow p = 2$. Focus: $F(0,2)$.

**(b)** $P(4,2)$, $F(0,2)$. Distance $PF = 4$.
Distance from $P$ to directrix $y = -2$: $2 + 2 = 4$. ✓
Reflected ray goes from $P$ toward $F$ (horizontal).

**(c)** $y' = \frac{x}{4}$. At $x = 4$, slope $= 1$.
Tangent: $y - 2 = 1(x - 4) \Rightarrow y = x - 2$.

**(d)** Incoming ray: vertical. Tangent slope $1$ (angle $45°$).
Outgoing toward $F$: horizontal.
Angle of incidence = angle of reflection = $45°$ with the tangent. ✓

## Problem 42 :

**Answer:**

**(a)** Circle centre $C$, radius $r$. Homothety centre $O$, ratio $k$.
$C' = O + k(C - O)$.

**(b)** For point $P$ on the circle: $\parallel PC \parallel = r$.
$P' = O + k(P - O)$.
$\parallel P'C' \parallel = \parallel O + k(P - O) - (O + k(C - O)) \parallel = \parallel k(P - C) \parallel = \mid k \mid \parallel P - C \parallel = \mid k \mid r$.

**(c)** So all images $P'$ lie on a circle with centre $C'$ and radius $\mid k \mid r$.

**(d)** Circle centre $(1,2)$, radius $3$, $O = (0,0)$, $k = 2$:
$C' = (2,4)$, $r' = 6$.
Image circle: centre $(2,4)$, radius $6$.

## Problem 43 :

**Answer:**

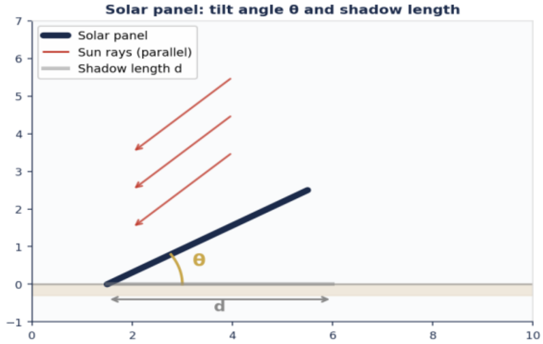
**(a)** If an isometry fixes two points $A$ and $B$, it fixes all points on line $AB$, or reflects across $AB$.

**(b)** An isometry fixing a single point $O$ is either a rotation or a reflection across a line through $O$.

**(c)** An isometry with no fixed points is either a translation or a glide reflection.

**(d)** Examples:
Translation: $(x, y) \rightarrow (x + 3, y)$.
Rotation: $90°$ about origin.
Reflection: across x-axis.
Glide reflection: reflect across x-axis then translate by $(2,0)$.

## Problem 44 :



**Answer:**

**(a)** Effective capture area $= A \cdot \cos(\theta - \alpha) = 2 \cdot \cos(25° - 35°) = 2\cos(-10°) \approx 2 \cdot 0.985 =$ 1.97 m².

**(b)** $\cos(\theta - \alpha)$ maximised when $\theta - \alpha = 0 \Rightarrow \theta^* = \alpha$. Optimal tilt equals sun elevation.

**(c)** $d = \frac{2\cos(25°)}{\sin(35°)} \approx \frac{2(0.906)}{0.574} \approx 3.16$ m.

**(d)** $d \leq 3 \Rightarrow \frac{2\cos\theta}{\sin\alpha} \leq 3 \Rightarrow \sin\alpha \geq \frac{2\cos(25°)}{3} \approx 0.604 \Rightarrow \alpha \geq \arcsin(0.604) \approx 37.2°$.

$$\boxed{\text{(a) } 1.97 \text{ m}^2 \text{(b) } \theta^* = \alpha \text{(c) } 3.16 \text{ m(d) } \alpha \geq 37.2°}$$

.

## Problem 45 :

**Answer:**

**(a)** Centre of equilateral triangle on side $BC$:
$G_a = \frac{B+C}{2} + \frac{1}{\sqrt{3}}$ times the perpendicular direction (outward).

**(b)** With complex numbers, $G_a = \frac{B+C}{2} + \frac{i}{\sqrt{3}}(C - B)$. Similar for $G_b, G_c$.

**(c)** For $A(0,0)$, $B(4,0)$, $C(2,3)$:
$G_a = \left(3,1.5 + \frac{2}{\sqrt{3}}\right) \approx (3,2.655)$.
Compute $G_b, G_c$ similarly.

**(d)** Numerically, $\| G_aG_b \| = \| G_bG_c \| = \| G_cG_a \|$ (all equal) → equilateral triangle.

## Problem 46 :

**Answer:**

**(a)** Rectangle $\phi \times 1$. Remove a $1 \times 1$ square: leaves $(\phi - 1) \times 1$.
$\phi - 1 = \frac{1}{\phi}$, so the ratio is $1/(\phi - 1) = \phi$. ✓

**(b)** $4\phi \approx 6.47$ m or $4/\phi \approx 2.47$ m.

**(c)** $\frac{F_{n+1}}{F_n} \to \phi$ (proved in Chapter 5).

**(d)** Radii: $1,1,2,3,5$ (Fibonacci numbers).

.

## Problem 47 :

**Answer:**

**(a)** Draw perpendiculars from $A, B, C$ to the transversal.
Let the signed distances be $h_A, h_B, h_C$.

**(b)** By similar triangles:
$\frac{BD}{DC} = -\frac{h_B}{h_C}, \frac{CE}{EA} = -\frac{h_C}{h_A}, \frac{AF}{FB} = -\frac{h_A}{h_B}$.

**(c)** Multiply: $\left( -\frac{h_B}{h_C}\right)\left( -\frac{h_C}{h_A}\right)\left( -\frac{h_A}{h_B}\right) = -1$. ✓

**(d)** For triangle $A(0,3)$, $B(-2,0)$, $C(4,0)$, transversal $x = 1$:
$D(1,0)$, $E(1,\frac{9}{4})$, $F(1,\frac{9}{2})$.
Check Menelaus with signed ratios.

## Problem 48 :

**Answer:**

**(a)** If $M$ is the midpoint of a chord through $P$, then $OM \perp$ the chord (radius to midpoint of chord is perpendicular).

**(b)** So $\angle OMP = 90°$. By Thales' theorem, $M$ lies on the circle with diameter $OP$.

**(c)** Centre = midpoint of $OP$, radius = $\frac{OP}{2}$.

**(d)** For circle $x^2 + y^2 = 25$, $P(2,0)$:
Locus: circle with centre $(1,0)$, radius $1$.
Equation: $(x - 1)^2 + y^2 = 1$.

## Problem 49 :

**Answer:**

This is a classic projective geometry result. A numerical verification:

**(a)** Unit circle $x^2 + y^2 = 1$, $PQ$ horizontal chord $y = 0.5$.
Intersections: $P = (-\frac{\sqrt{3}}{2}, 0.5)$, $Q = (\frac{\sqrt{3}}{2}, 0.5)$.
Midpoint $M = (0,0.5)$.

**(b)** Chord $AB$ through $M$ with slope 2: $y = 2x + 0.5$.
Intersections with circle give $A \approx (0.235,0.97)$, $B \approx (-0.635, -0.77)$.

**(c)** Chord $CD$ through $M$ with slope $0.5$: $y = 0.5x + 0.5$.
Intersections give $C$, $D$.

**(d)** Compute $X = AD \cap PQ$, $Y = BC \cap PQ$. Verify midpoint is $M$. ✓

## Problem 50 :

**Answer:**

**(a)** Circumcentre $O$: perpendicular bisector of $AB$: $x = 4$.
Perpendicular bisector of $AC$: midpoint $(2,3)$, slope $AC = \frac{6}{4} = \frac{3}{2}$, perpendicular slope $-\frac{2}{3}$.
Line: $y - 3 = -\frac{2}{3}(x - 2)$. At $x = 4$: $y = -\frac{8}{3} + 3 = \frac{1}{3}$.
$O = \left(4,\frac{1}{3}\right)$.

**(b)** $R = \| OA \| = \sqrt{16 + \frac{1}{9}} = \sqrt{\frac{145}{9}} = \frac{\sqrt{145}}{3} \approx 4.01$.

**(c)** Check if $O$ is inside triangle: base $AB$ is y=0, apex C is (4,6).
At x=4, O has y=1/3, which is between 0 and 6 → inside.

**(d)** The Fermat point (minimising sum of distances) for an acute triangle is the point where all three sides subtend 120°.
Check angles: all acute → Fermat point exists.

$$\boxed{\text{(a) } O = \left(4,\frac{1}{3}\right) \text{ (b) } R = \frac{\sqrt{145}}{3} \text{ (c) Inside}}$$

CHAPTER 9 — SPACE GEOMETRY

# SOLUTIONS OF PROBLEMS

## Problems 1–15 — Instant Recall

### Problem 1 :

**Answer:** **(a)** $\overrightarrow{AB} = (4-1,\ -1-2,\ 3-(-1)) = (3,-3,4)$.
$\| AB \| = \sqrt{3^2+(-3)^2+4^2} = \sqrt{9+9+16} = \sqrt{34}$.
**(b)** $\overrightarrow{OM} = (3-0,\ 4-0,\ 0-0) = (3,4,0)$.
$\| OM \| = \sqrt{3^2+4^2+0^2} = \sqrt{9+16} = 5$.
**(c)** $\overrightarrow{PQ} = (2-2,\ 1-1,\ -1-5) = (0,0,-6)$.
$\| PQ \| = \sqrt{0+0+36} = 6$.
**(d)** $\overrightarrow{AB} = (2-(-1),\ -1-3,\ 2-2) = (3,-4,0)$.
$\| AB \| = \sqrt{3^2+(-4)^2+0^2} = \sqrt{9+16} = 5$.

$$\boxed{\text{(a) } \sqrt{34} \text{(b) } 5 \text{(c) } 6 \text{(d) } 5}$$

### Problem 2 :

**Answer:**

The midpoint formula in 3D is: $M = \left(\frac{x_1+x_2}{2},\ \frac{y_1+y_2}{2},\ \frac{z_1+z_2}{2}\right)$.
**(a)** $M = \left(\frac{2+8}{2},\ \frac{4+0}{2},\ \frac{6+2}{2}\right) = (5,2,4)$.
**(b)** $M = \left(\frac{-1+5}{2},\ \frac{3-3}{2},\ \frac{5+1}{2}\right) = (2,0,3)$.
**(c)** $M = \left(\frac{0+a}{2},\ \frac{0+b}{2},\ \frac{0+c}{2}\right) = \left(\frac{a}{2},\ \frac{b}{2},\ \frac{c}{2}\right)$.

$$\boxed{\text{(a) } (5,2,4) \text{(b) } (2,0,3) \text{(c) } \left(\frac{a}{2},\frac{b}{2},\frac{c}{2}\right)}$$

### Problem 3 :

**Answer:**

**(a)** $\overrightarrow{AB} = (4-1,\ 3-0,\ 1-(-2)) = (3,3,3)$.
**(b)** $\overrightarrow{AB} = (0-3,\ 0-2,\ 0-1) = (-3,-2,-1)$.
**(c)** $\overrightarrow{AB} = (5-2,\ -1-(-1),\ 4-4) = (3,0,0)$.

$$\boxed{\text{(a) } (3,3,3) \text{(b) } (-3,-2,-1) \text{(c) } (3,0,0)}$$

### Problem 4 :

**Answer:**

**(a)** $u \cdot v = 1(-1)+2(1)+3(-1) = -1+2-3 = -2 \neq 0$.
**Not orthogonal.**
**(b)** $u \cdot v = 2(1)+0(3)+(-1)(2) = 2+0-2 = 0$.
**Orthogonal.**
**(c)** $u \cdot v = 1(1)+1(-2)+1(1) = 1-2+1 = 0$.
**Orthogonal.**

$$\boxed{\begin{array}{c}\text{(a) } -2, \text{ not orthogonal(b) } 0, \text{ orthogonal} \\ \text{(c) } 0, \text{ orthogonal}\end{array}}$$

### Problem 5 :

**Answer:**

The equation of a plane with normal $n = (a,b,c)$ through point $M(x_0,y_0,z_0)$ is:
$a(x-x_0)+b(y-y_0)+c(z-z_0) = 0$.
**(a)** $1(x-3)+2(y-0)-1(z-1) = 0 \Rightarrow x-3+2y-z+1 = 0 \Rightarrow x+2y-z-2 = 0 \Rightarrow x+2y-z = 2$.
**(b)** $0(x-0)+0(y-0)+1(z-5) = 0 \Rightarrow z-5 = 0 \Rightarrow z = 5$.
**(c)** $2(x-1)-1(y-1)+3(z-0) = 0 \Rightarrow 2x-2-y+1+3z = 0 \Rightarrow 2x-y+3z-1 = 0 \Rightarrow 2x-y+3z = 1$.

$$\boxed{\text{(a) } x+2y-z=2 \text{(b) } z=5 \text{(c) } 2x-y+3z=1}$$

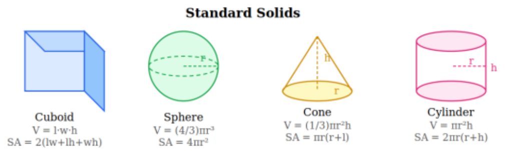
### Problem 6 :

**Answer:**

The parametric equations of a line through $A(x_0,y_0,z_0)$ with direction $u = (a,b,c)$ are:
$x = x_0+at,\ y = y_0+bt,\ z = z_0+ct$.
**(a)** $x = 1+2t,\ y = 2-t,\ z = 3+t$.
**(b)** $x = t,\ y = t,\ z = t$.
**(c)** $x = 3+0t = 3,\ y = 1+t,\ z = -2+0t = -2$.
This is a line parallel to the $y$-axis passing through $(3,\_,-2)$.

$$\boxed{\begin{array}{c}\text{(a) } x=1+2t,\ y=2-t,\ z=3+t \\ \text{(b) } x=y=z=t \\ \text{(c) } x=3,\ y=1+t,\ z=-2\end{array}}$$

### Problem 7 :



**Answer:**

**(a)** $V = 3 \times 5 \times 4 = 60$ cm³.
**(b)** $V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi \cdot 6^3 = \frac{4}{3}\pi \cdot 216 = 288\pi \approx$ 905 cm³.
**(c)** $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \cdot 3^2 \cdot 8 = 24\pi \approx 75.4$ cm³.
**(d)** $V = \pi r^2 h = \pi \cdot 2^2 \cdot 10 = 40\pi \approx 125.7$ cm³.

$$\boxed{\text{(a) } 60 \text{ cm}^3 \text{(b) } 288\pi \text{ cm}^3 \text{(c) } 24\pi \text{ cm}^3 \text{(d) } 40\pi \text{ cm}^3}$$

### Problem 8 :

**Answer:**

**(a)** $A = 6a^2 = 6 \cdot 5^2 = 150$ cm².
**(b)** $A = 4\pi r^2 = 4\pi \cdot 4^2 = 64\pi \approx 201$ cm².
**(c)** $A = 2\pi r(r+h) = 2\pi \cdot 3(3+7) = 60\pi \approx$ 188.5 cm².
**(d)** $A = \pi r(r+l) = \pi \cdot 2(2+5) = 14\pi \approx 44$ cm².

$$\boxed{\text{(a) } 150 \text{ cm}^2 \text{(b) } 64\pi \text{ cm}^2 \text{(c) } 60\pi \text{ cm}^2 \text{(d) } 14\pi \text{ cm}^2}$$

### Problem 9 :

**Answer:**

**(a)** $P_2 = 2P_1$ (multiply the first equation by 2).
They have the same normal and the same set of points.
**Equal planes.**

**(b)** Both have normal $n = (1,1,1)$ but different constants (1 vs 2).
**Parallel distinct planes (no intersection).**
**(c)** Normals are $n_1 = (1,1,0)$ and $n_2 = (0,1,1)$. They are not scalar multiples, so the planes are not parallel.
**Intersecting along a line.**

$$\boxed{\text{(a) Equal(b) Parallel(c) Intersecting}}$$

## Problem 10 :

**Answer:**

**(a)** From $x = 3$: $1 + 2t = 3 \Rightarrow t = 1$. Then $y = 1 - 3 = -2$, but the point has $y = 0$.
**Not on the line.**
**(b)** Substitute $Q$: $3(1) - 2 + 2(1) = 3 - 2 + 2 = 3 \neq 5$.
**Not on the plane.**
**(c)** Substitute $R$: $2 + 2(1) - (-1) = 2 + 2 + 1 = 5 \neq 7$.
**Not on the plane.**

$$\boxed{\text{(a) No(b) No(c) No}}$$

## Problem 11 :

**Answer:**

**(a)** $(x-2)^2 + (y+1)^2 + (z-3)^2 = 5^2 \Rightarrow (x-2)^2 + (y+1)^2 + (z-3)^2 = 25$.
**(b)** Radius $r = \| OA \| = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{9} = 3$.
Equation: $x^2 + y^2 + z^2 = 9$.
**(c)** Centre $M$ = midpoint of $AB$ = $\left(\frac{0+4}{2}, \frac{0+2}{2}, \frac{0+2}{2}\right) = (2,1,1)$.
Radius $r = \| MA \| = \sqrt{(2-0)^2 + (1-0)^2 + (1-0)^2} = \sqrt{4+1+1} = \sqrt{6}$.
Equation: $(x-2)^2 + (y-1)^2 + (z-1)^2 = 6$.

$$\boxed{\begin{array}{c}\text{(a) } (x-2)^2 + (y+1)^2 + (z-3)^2 = 25 \\ \text{(b) } x^2 + y^2 + z^2 = 9 \\ \text{(c) } (x-2)^2 + (y-1)^2 + (z-1)^2 = 6\end{array}}$$

## Problem 12 :

**Answer:**

**(a)** $n_a = (3,-1,2)$.
**(b)** $n_b = (1,3,0)$.
**(c)** The second plane has normal $n_c = (6,-2,4) = 2(3,-1,2) = 2n_a$. Since the normals are parallel, the planes are **parallel**.
**(d)** The plane $x + y + z = 1$ has normal $n_d = (1,1,1)$.
$n_a \cdot n_d = 3(1) + (-1)(1) + 2(1) = 3 - 1 + 2 = 4 \neq 0$.
Since the dot product is not zero, the normals are not perpendicular, so the planes are **not perpendicular**.

$$\boxed{\text{(a) } (3,-1,2)\text{(b) } (1,3,0)\text{(c) Parallel(d) Not perpendicular}}$$

## Problem 13 :

**Answer:**

**(a)** $l = \sqrt{r^2 + h^2} = \sqrt{25 + 144} = \sqrt{169} = 13$ cm.
**(b)** Lateral area $= \pi r l = \pi \cdot 5 \cdot 13 = 65\pi \approx 204$ cm².
**(c)** Total area $= \pi r(r + l) = \pi \cdot 5(5 + 13) = 90\pi \approx 283$ cm².
**(d)** Volume $= \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \cdot 25 \cdot 12 = 100\pi \approx 314$ cm³.

$$\boxed{\text{(a) 13 cm(b) } 65\pi \text{ cm}^2\text{(c) } 90\pi \text{ cm}^2\text{(d) } 100\pi \text{ cm}^3}$$

## Problem 14 :

**Answer:**

**(a)** Substitute the parametric equations into the plane:
$(1 + t) + 2(2 - t) - (3t) = 4 \Rightarrow 1 + t + 4 - 2t - 3t = 4 \Rightarrow 5 - 4t = 4 \Rightarrow -4t = -1 \Rightarrow t = \frac{1}{4}$.
Point: $\left(1 + \frac{1}{4}, 2 - \frac{1}{4}, 3 \cdot \frac{1}{4}\right) = \left(\frac{5}{4}, \frac{7}{4}, \frac{3}{4}\right)$.
**(b)** Substitute: $t + t + t = 6 \Rightarrow 3t = 6 \Rightarrow t = 2$.
Point: $(2, 2, 2)$.

$$\boxed{\text{(a) } \left(\frac{5}{4}, \frac{7}{4}, \frac{3}{4}\right)\text{(b) } (2,2,2)}$$

## Problem 15 :

**Answer:**

**(a)** $a = 1,\ b = 2,\ c = -2,\ d = 1$. Evaluate at $M(1,2,3)$:
$1(1) + 2(2) - 2(3) + 1 = 1 + 4 - 6 + 1 = 0$.
$d = \frac{|0|}{\sqrt{1+4+4}} = 0$.
The point $M$ lies on the plane.
**(b)** Rewrite: $3x - 4y - 15 = 0$ (so $a = 3,\ b = -4,\ c = 0,\ d = -15$). At $O(0,0,0)$:
$d = \frac{|0+0+0-15|}{\sqrt{9+16}} = \frac{15}{5} = 3$.
**(c)** Rewrite: $2x + y - 2z - 9 = 0$ (so $a = 2,\ b = 1,\ c = -2,\ d = -9$). At $P(0,0,0)$:
$d = \frac{|0+0+0-9|}{\sqrt{4+1+4}} = \frac{9}{\sqrt{9}} = 3$.

$$\boxed{\text{(a) 0 (on the plane)(b) 3(c) 3}}$$

# Problems 16–38 — Medium

## Problem 16 :

**Answer:**

The cross product formula is:

$$u \times v = (u_y v_z - u_z v_y,\ u_z v_x - u_x v_z,\ u_x v_y - u_y v_x).$$
$$n_x = 1 \cdot 2 - (-1) \cdot 3 = 2 + 3 = 5.$$
$$n_y = (-1) \cdot 1 - 2 \cdot 2 = -1 - 4 = -5.$$
$$n_z = 2 \cdot 3 - 1 \cdot 1 = 6 - 1 = 5.$$

Thus $n = (5,-5,5)$. This can be simplified by dividing by 5:

$$n = (1,-1,1).$$

$$\boxed{u \times v = (1,-1,1)}$$

## Problem 17 :

**Answer:**

**(a)** Compute vectors in the plane:

$$\overrightarrow{AB} = (3 - 1,\ 1 - 0,\ 0 - 2) = (2,1,-2),$$
$$\overrightarrow{AC} = (0 - 1,\ 2 - 0,\ 1 - 2) = (-1,2,-1).$$

**(b)** Find the normal vector:

$$n = \overrightarrow{AB} \times \overrightarrow{AC}.$$
$$n_x = 1 \cdot (-1) - (-2) \cdot 2 = -1 + 4 = 3.$$
$$n_y = (-2) \cdot (-1) - 2 \cdot (-1) = 2 + 2 = 4.$$
$$n_z = 2 \cdot 2 - 1 \cdot (-1) = 4 + 1 = 5.$$

So $n = (3,4,5)$.

**(c)** Using point $A(1,0,2)$:

$3(x-1) + 4(y-0) + 5(z-2) = 0.$

$3x - 3 + 4y + 5z - 10 = 0 \Rightarrow 3x + 4y + 5z = 13.$

$\boxed{3x + 4y + 5z = 13}$

## Problem 18 :

**Answer:** Lines $L_1$ passes through $A(1,0,1)$ with direction $\mathrm{u} = (1,1,0)$.

$L_2$ passes through $B(0,1,0)$ with direction $\mathrm{v} = (1,0,1)$.

(a) Are the lines parallel?

(b) Do they intersect? If so, find the intersection point.

(c) State their relative position (parallel, intersecting, or skew).

Corrected full answer:

(a) Two lines are parallel if their direction vectors are scalar multiples.

$\mathrm{u} = (1,1,0)$ and $\mathrm{v} = (1,0,1)$.

There is no scalar $\lambda$ such that $(1,1,0) = \lambda(1,0,1)$ (the second component would require $1 = \lambda \cdot 0$, impossible).

Answer: $\boxed{\text{Not parallel}}$

(b) Solve $A + t\mathrm{u} = B + s\mathrm{v}$:

$1 + t = s$ … (1)

$0 + t = 1 + 0 \cdot s \Rightarrow t = 1$ … (2)

$1 + 0 = 0 + s \Rightarrow s = 1$ … (3)

Substitute $t = 1, s = 1$ into (1): $1 + 1 = 1 \Rightarrow 2 = 1$, contradiction.

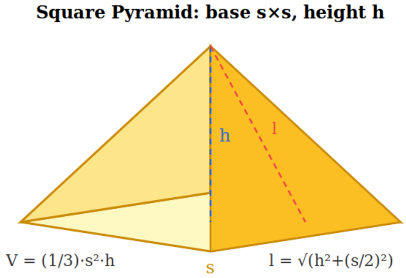
No solution.

Answer: $\boxed{\text{No intersection}}$

(c) The lines are not parallel and do not intersect. Therefore they are skew lines.

Answer: $\boxed{\text{Skew}}$

## Problem 19 :



**Answer:**

**(a)** Base area: $6 \times 6 = 36$ cm².

**(b)** Volume: $V = \frac{1}{3} \times \text{Base Area} \times h = \frac{1}{3} \times 36 \times 9 = 108$ cm³.

**(c)** The distance from the centre of the square base to the midpoint of a side is half the side length: 3 cm.

Slant height: $l = \sqrt{9^2 + 3^2} = \sqrt{81 + 9} = \sqrt{90} = 3\sqrt{10}$ cm.

**(d)** Lateral surface area consists of 4 identical triangles, each with base 6 cm and height $l = 3\sqrt{10}$ cm:

$$A_{\text{lateral}} = 4 \cdot \frac{1}{2} \cdot 6 \cdot 3\sqrt{10} = 12 \cdot 3\sqrt{10} = 36\sqrt{10} \approx 113.8 \text{ cm}^2.$$

$\boxed{\text{(a) } 36 \text{ cm}^2 \text{(b) } 108 \text{ cm}^3 \text{(c) } 3\sqrt{10} \text{ cm(d) } 36\sqrt{10} \text{ cm}^2}$

## Problem 20 :

**Answer:**

**(a)** $r = \| CP \| = \sqrt{(4-1)^2 + (-2+2)^2 + (3-3)^2} = \sqrt{3^2 + 0 + 0} = 3.$

**(b)** Equation: $(x-1)^2 + (y+2)^2 + (z-3)^2 = 9.$

**(c)** For $Q(1,1,3)$:

$$CQ^2 = (1-1)^2 + (1+2)^2 + (3-3)^2 = 0 + 9 + 0 = 9 = r^2.$$

Thus $Q$ is **on the sphere**.

$\boxed{\begin{array}{c}\text{(a) } 3 \text{(b) } (x-1)^2 + (y+2)^2 + (z-3)^2 = 9 \\ \text{(c) On the sphere}\end{array}}$

## Problem 21 :

**Answer:**

**(a)** Normals: $n_1 = (1,1,1)$, $n_2 = (2,-1,1)$.

Direction vector of the line of intersection:

$$\begin{aligned} d = n_1 \times n_2 &= (1 \cdot 1 - 1 \cdot (-1),\ 1 \cdot 2 - 1 \cdot 1,\ 1 \cdot (-1) - 1 \cdot 2) \\ &= (1+1,\ 2-1,\ -1-2) \\ &= (2,1,-3). \end{aligned}$$

**(b)** Find a point on the line by setting $z = 0$:

$$x + y = 3, 2x - y = 1.$$

Add: $3x = 4 \Rightarrow x = \frac{4}{3}$. Then $y = 3 - \frac{4}{3} = \frac{5}{3}$.

Point: $\left(\frac{4}{3}, \frac{5}{3}, 0\right)$.

**(c)** Parametric equations:

$$x = \frac{4}{3} + 2t, y = \frac{5}{3} + t, z = -3t.$$

$\boxed{x = \frac{4}{3} + 2t,\ y = \frac{5}{3} + t,\ z = -3t}$

## Problem 22 :

**Answer:**

The angle between planes equals the angle between their normals:

$$\cos\theta = \frac{| n_1 \cdot n_2 |}{\| n_1 \| \| n_2 \|}.$$

**(a)** $n_1 = (1,1,1)$, $n_2 = (0,0,1)$.

$n_1 \cdot n_2 = 1$, $\| n_1 \| = \sqrt{3}$, $\| n_2 \| = 1$.

$\cos\theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = \arccos\left(\frac{1}{\sqrt{3}}\right) \approx 54.7°$.

**(b)** $n_1 = (1,-1,0)$, $n_2 = (0,1,-1)$.

$| n_1 \cdot n_2 | = | 0 - 1 + 0 | = 1.$

$\| n_1 \| = \sqrt{2}$, $\| n_2 \| = \sqrt{2}$.

$\cos\theta = \frac{1}{2} \Rightarrow \theta = 60°$.

$\boxed{\text{(a) } 54.7° \text{(b) } 60°}$

## Problem 23 :

**Answer:**

**(a)** Substitute line into plane:
$(1+t)+(3-t)=4 \Rightarrow 4=4$ for all $t$.
**Line is contained in the plane.**
**(b)** Substitute: $t+2t+3t=6t=0 \Rightarrow t=0$.
**Line intersects the plane at the origin.**
**(c)** Substitute: $(1+2t)+(1+t)-(1+3t)=1+2t+1+t-1-3t=1$ for all $t$.
**Line is contained in the plane.**

$\boxed{\text{(a) Contained(b) Intersects at origin(c) Contained}}$

## Problem 24 :

**Answer:**

Normal: $n=(2,-1,2)$, $\| n \|^2=4+1+4=9$.
$n \cdot P=2(3)-1(1)+2(4)=6-1+8=13$.
The plane has $d=0$. The foot is:

$$H=P-\frac{n\cdot P+d}{\| n \|^2}n=P-\frac{13}{9}(2,-1,2).$$

$$H=\left(3-\frac{26}{9},\ 1+\frac{13}{9},\ 4-\frac{26}{9}\right)$$
$$=\left(\frac{27-26}{9},\ \frac{9+13}{9},\ \frac{36-26}{9}\right)$$
$$=\left(\frac{1}{9},\ \frac{22}{9},\ \frac{10}{9}\right).$$

$$\boxed{H=\left(\frac{1}{9},\ \frac{22}{9},\ \frac{10}{9}\right)}$$

## Problem 25 :

**Answer:**

The volume of a tetrahedron with vertices $O, A, B, C$ is:

$$V=\frac{1}{6}\left|\det\begin{pmatrix}2&0&0\\0&3&0\\0&0&4\end{pmatrix}\right|.$$

The determinant is $2\cdot 3\cdot 4=24$.
$V=\frac{24}{6}=4$ cubic units.

$$\boxed{4}$$

## Problem 26 :

**Answer:**

Complete the square in each variable:

$$(x^2-4x)+(y^2+6y)+(z^2-2z)=2.$$
$$(x-2)^2-4+(y+3)^2-9+(z-1)^2-1=2.$$
$$(x-2)^2+(y+3)^2+(z-1)^2=2+4+9+1$$
$$=16.$$

Centre: $(2,-3,1)$, radius $r=4$.

$$\boxed{(x-2)^2+(y+3)^2+(z-1)^2=16}$$

## Problem 27 :

**Answer:**

**(a)** Same normal: $n=(2,-1,3)$.
$2(x-1)-(y-2)+3(z-3)=0 \Rightarrow 2x-2-y+2+3z-9=0 \Rightarrow 2x-y+3z=9$.
**(b)** The direction of the line is $(2,-1,3)$, which becomes the normal: same plane: $2x-y+3z=9$.
**(c)** Check $M(1,2,3)$: $2(1)-2+3(3)=2-2+9=9$ ✓.

$$\boxed{2x-y+3z=9}$$

## Problem 28 :

**Answer:**

**(a)** $\overrightarrow{AB}=(-1,1,0)$, $\overrightarrow{AC}=(-1,0,1)$.

$$\overrightarrow{AB}\times\overrightarrow{AC}=(1\cdot 1-0\cdot 0,\ 0\cdot(-1)-(-1)\cdot 1,\ (-1)\cdot 0-1\cdot(-1))=(1,\ 1,\ 1).$$

**(b)** Area $=\frac{1}{2}\| \overrightarrow{AB}\times\overrightarrow{AC} \|=\frac{1}{2}\sqrt{1^2+1^2+1^2}=\frac{\sqrt{3}}{2}$.
**(c)** Unit normal: $\frac{(1,1,1)}{\sqrt{3}}$.
**(d)** Plane through $A(1,0,0)$: $1(x-1)+1(y-0)+1(z-0)=0 \Rightarrow x+y+z=1$.

$$\boxed{\text{(a) }(1,1,1)\text{(b) }\frac{\sqrt{3}}{2}\text{(c) }\frac{(1,1,1)}{\sqrt{3}}\text{(d) }x+y+z=1}$$

## Problem 29 :

**Answer:**

The line passes through $A(0,0,0)$ with direction $u=(1,1,1)$.

$$\overrightarrow{AP}=(1,2,3).$$
$$\overrightarrow{AP}\times u=(2\cdot 1-3\cdot 1,\ 3\cdot 1-1\cdot 1,\ 1\cdot 1-2\cdot 1)$$
$$=(-1,\ 2,\ -1).$$
$$\| \overrightarrow{AP}\times u \|=\sqrt{1+4+1}=\sqrt{6}.$$
$$\| u \|=\sqrt{1+1+1}=\sqrt{3}.$$

Distance: $d=\frac{\|\overrightarrow{AP}\times u\|}{\|u\|}=\frac{\sqrt{6}}{\sqrt{3}}=\sqrt{2}$.

$$\boxed{\sqrt{2}}$$

## Problem 30 :

**Answer:**

**(a)** Cylinder: $V_c=\pi r^2h=\pi\cdot 4\cdot 5=20\pi\approx 62.8$ m³.
Sphere: $V_s=\frac{4}{3}\pi r^3=\frac{4}{3}\pi\cdot 8=\frac{32\pi}{3}\approx 33.5$ m³.
The **cylinder holds more**.
**(b)** Set cylinder volume equal to sphere volume:
$\pi\cdot 4\cdot h=\frac{32\pi}{3}\Rightarrow 4h=\frac{32}{3}\Rightarrow h=\frac{8}{3}\approx 2.67$ m.
**(c)** Difference in volume: $20\pi-\frac{32\pi}{3}=\frac{60\pi-32\pi}{3}=\frac{28\pi}{3}$ m³.
Cost difference: $\frac{28\pi}{3}\cdot 500\approx 14660$ DA.

$$\boxed{\text{(a) Cylinder(b) 2.67 m(c) 14660 DA}}$$

## Problem 31 :

**Answer:**

**(a)** Using intercept form: $\frac{x}{3}+\frac{y}{4}+\frac{z}{6}=1$.
Multiply by 12: $4x+3y+2z=12$.
**(b)** The tetrahedron formed by intercepts $a=3$, $b=4$, $c=6$ has volume:
$V=\frac{1}{6}abc=\frac{1}{6}\cdot 3\cdot 4\cdot 6=\frac{72}{6}=12$.

$$\boxed{\text{(a) }4x+3y+2z=12\text{(b) }12}$$

## Problem 32 :

**Answer:**

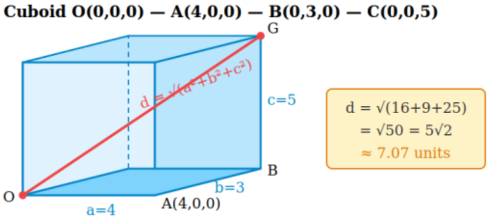
Compute $\overrightarrow{AB}=(-1,1,0)$, $\overrightarrow{AC}=(-1,0,1)$, $\overrightarrow{AD}=(0,1,1)$.
The four points are coplanar iff the scalar triple product is zero:

$$\det\begin{pmatrix}-1 & 1 & 0\\ -1 & 0 & 1\\ 0 & 1 & 1\end{pmatrix} = -1(0\cdot 1-1\cdot 1)-1(-1\cdot 1$$
$$-1\cdot 0)+0=-1(-1)-1(-1)$$
$$=1+1=2\neq 0.$$

Thus the points are **not coplanar**.

$\boxed{\text{Not coplanar}}$

## Problem 33 :


Cuboid O(0,0,0) — A(4,0,0) — B(0,3,0) — C(0,0,5)

### Answer:

**(a)** Vertices:

$O(0,0,0),\ A(4,0,0),\ B(0,3,0),\ C(0,0,5),$
$AB=(4,3,0),\ AC=(4,0,5),\ BC=(0,3,5),\ ABC=(4,3,5).$

**(b)** Main diagonal from $O$ to $(4,3,5)$:
$OG=\sqrt{4^2+3^2+5^2}=\sqrt{16+9+25}=\sqrt{50}=5\sqrt{2}.$

**(c)** The angle $\theta$ between the diagonal and the base plane is given by:
$\sin\theta=\frac{z\text{-component}}{\|OG\|}=\frac{5}{5\sqrt{2}}=\frac{1}{\sqrt{2}}\Rightarrow\theta=45^\circ.$

$$\boxed{\begin{array}{c}\text{(a) }(0,0,0),(4,0,0),(0,3,0),(0,0,5),\\(4,3,0),(4,0,5),(0,3,5),(4,3,5)\\\text{(b) }5\sqrt{2}\quad\text{(c) }45^\circ\end{array}}$$

## Problem 34 :

### Answer:

Centre of sphere: $C(1,2,3)$. Normal vector: $\overrightarrow{CP}=P-C=(2,2,3)$.

The tangent plane has normal $\overrightarrow{CP}$ and passes through $P$:
$$2(x-3)+2(y-4)+3(z-6)=0.$$
$$2x-6+2y-8+3z-18=0\Rightarrow 2x+2y+3z=32.$$

$\boxed{2x+2y+3z=32}$

## Problem 35 :

### Answer:

**(a)** Disk method around x-axis:
$$V=\pi\int_0^4(\sqrt{x})^2dx=\pi\int_0^4 x\,dx=\pi\left[\frac{x^2}{2}\right]_0^4=\pi\cdot\frac{16}{2}=8\pi.$$

**(b)** Shell method around y-axis:
$$V=2\pi\int_0^4 x\cdot\sqrt{x}dx=2\pi\int_0^4 x^{3/2}\,dx=2\pi\left[\frac{2}{5}x^{5/2}\right]_0^4=2\pi\cdot\frac{2}{5}\cdot 4^{5/2}=\frac{4\pi}{5}\cdot 32=\frac{128\pi}{5}.$$

$\boxed{\text{(a) }8\pi\text{(b) }\frac{128\pi}{5}}$

## Problem 36 :

### Answer:

**(a)** $v=2u$, so the lines are **parallel** ✓.

**(b)** $\overrightarrow{AB}=(2,2,2).$
$$u\times\overrightarrow{AB}=(2,1,-1)\times(2,2,2)=(1\cdot 2-(-1)\cdot 2,\ (-1)\cdot 2-2\cdot 2,\ 2\cdot 2-1\cdot 2)=(2+2,\ -2-4,\ 4-2)=(4,-6,2).$$
$$\|u\times\overrightarrow{AB}\|=\sqrt{16+36+4}=\sqrt{56}.$$
$$\|u\|=\sqrt{4+1+1}=\sqrt{6}.$$

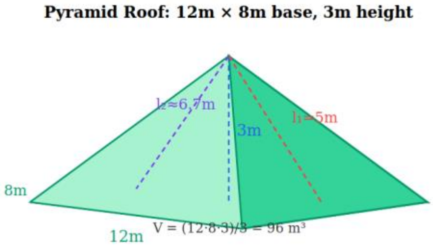
Distance: $d=\frac{\sqrt{56}}{\sqrt{6}}=\sqrt{\frac{56}{6}}=\sqrt{\frac{28}{3}}\approx 3.05.$

**(c)** The normal to the plane containing both lines is $n=u\times\overrightarrow{AB}=(4,-6,2)$ (or simplified $(2,-3,1)$).
Plane through $A(1,2,3)$:
$2(x-1)-3(y-2)+1(z-3)=0\Rightarrow 2x-3y+z=-1.$

(Check: $2(1)-3(2)+3=2-6+3=-1$).

$\boxed{\text{(a) Parallel(b) }\sqrt{\frac{28}{3}}\text{(c) }2x-3y+z=-1}$

## Problem 37 :


Pyramid Roof: 12m × 8m base, 3m height

### Answer:

**(a)** Base area = $12\times 8=96$ m².
$V=\frac{1}{3}\times 96\times 3=96$ m³.

**(b)** Half of the 8 m side is 4 m. Slant height:
$l_1=\sqrt{3^2+4^2}=\sqrt{9+16}=5$ m.

**(c)** Half of the 12 m side is 6 m. Slant height:
$l_2=\sqrt{3^2+6^2}=\sqrt{9+36}=\sqrt{45}=3\sqrt{5}\approx 6.7$ m.

**(d)** Two triangles with base 12 m and height 5 m, and two with base 8 m and height $3\sqrt{5}$ m:
$$A=2\cdot\frac{1}{2}\cdot 12\cdot 5+2\cdot\frac{1}{2}\cdot 8\cdot 3\sqrt{5}=60+24\sqrt{5}\approx 113.7\text{ m}^2.$$

$\boxed{\text{(a) }96\text{ m}^3\text{(b) }5\text{ m(c) }3\sqrt{5}\text{ m(d) }60+24\sqrt{5}\text{ m}^2}$

## Problem 38 :

### Answer:

**(a)** $x^2+y^2+z^2=25.$

**(b)** From B: $(x-10)^2+y^2+z^2=36.$
From C: $x^2+(y-10)^2+z^2=49.$

**(c)** Subtract (a) from (b):
$(x-10)^2-x^2+y^2-y^2+z^2-z^2=36-25$
$x^2-20x+100-x^2=11\Rightarrow -20x+100=11\Rightarrow -20x=-89\Rightarrow x=\frac{89}{20}=4.45.$

Subtract (a) from (c):

$$x^2 - x^2 + (y-10)^2 - y^2 + z^2 - z^2 = 49 - 25$$

$y^2 - 20y + 100 - y^2 = 24 \Rightarrow -20y + 100 = 24 \Rightarrow -20y = -76 \Rightarrow y = \frac{76}{20} = 3.8$.

Now from (a) with $z = 0.5$:

$4.45^2 + 3.8^2 + 0.5^2 = 19.8025 + 14.44 + 0.25 = 34.4925 \neq 25$.

The system is inconsistent (the three spheres do not all meet at the same point at $z = 0.5$).

In real GPS, an over-determined system (using time of arrival) is solved.

$$\boxed{\begin{gathered} x = 4.45,\ y = 3.8,\ z = 0.5 \text{ does not satisfy} \\ \text{all equations; system inconsistent} \end{gathered}}$$

# Problems 39–50 — Difficult & Real-World

## Problem 39 :

**Answer:**

Place a cuboid with vertices at $O(0,0,0)$ and opposite vertex $G(a,b,c)$. The four space diagonals are:

- $OG$
- $AD$ where $A = (a,0,0)$, $D = (0,b,c)$
- $BE$ where $B = (0,b,0)$, $E = (a,0,c)$
- $CF$ where $C = (0,0,c)$, $F = (a,b,0)$

**(a)** Midpoint of $OG$: $\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)$.

**(b)** Midpoint of $AD$: $\left(\frac{a+0}{2}, \frac{0+b}{2}, \frac{0+c}{2}\right) = \left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)$.

**(c)** By symmetry, all four diagonals have the same midpoint $\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)$.

Therefore, the four diagonals bisect each other at this point.

$$\boxed{\text{All diagonals bisect each other at } \left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)}$$

## Problem 40 :

**Answer:**

The general equation of a sphere is: $x^2 + y^2 + z^2 + Dx + Ey + Fz + G = 0$.

**(a)** Substitute $O(0,0,0)$: $G = 0$.

**(b)** Substitute $A(4,0,0)$: $16 + 4D = 0 \Rightarrow D = -4$.

Substitute $B(0,6,0)$: $36 + 6E = 0 \Rightarrow E = -6$.

Substitute $C(0,0,2)$: $4 + 2F = 0 \Rightarrow F = -2$.

**(c)** The sphere is: $x^2 + y^2 + z^2 - 4x - 6y - 2z = 0$.

Complete the square:

$$(x-2)^2 - 4 + (y-3)^2 - 9 + (z-1)^2 - 1 = 0$$
$$(x-2)^2 + (y-3)^2 + (z-1)^2 = 14.$$

Centre: $(2,3,1)$, radius: $r = \sqrt{14}$.

$$\boxed{(x-2)^2 + (y-3)^2 + (z-1)^2 = 14}$$

## Problem 41 :

**Answer: (a)** Planes are parallel if their normals are proportional:

$$\frac{2}{b} = \frac{-1}{3} = \frac{a}{-6}.$$

From $\frac{-1}{3} = \frac{a}{-6}$: $a = 2$.

From $\frac{2}{b} = \frac{-1}{3}$: $b = -6$.

**(b)** With $a = 3$, $b = 1$: $P: 2x - y + 3z = 5$, $Q: x + 3y - 6z = 2$.

Normals: $n_1 = (2,-1,3)$, $n_2 = (1,3,-6)$.

Direction vector:

$$\begin{aligned} d = n_1 \times n_2 &= (-1 \cdot (-6) - 3 \cdot 3,\ 3 \cdot 1 - 2 \\ &\quad \cdot (-6),\ 2 \cdot 3 - (-1) \cdot 1) \\ &= (6 - 9,\ 3 + 12,\ 6 + 1) \\ &= (-3,15,7). \end{aligned}$$

Set $z = 0$: $2x - y = 5$, $x + 3y = 2$. Solve:

From second: $x = 2 - 3y$. Substitute: $2(2 - 3y) - y = 5 \Rightarrow 4 - 6y - y = 5 \Rightarrow -7y = 1 \Rightarrow y = -\frac{1}{7}$, $x = 2 + \frac{3}{7} = \frac{17}{7}$.

Line: $\left(\frac{17}{7} - 3t,\ -\frac{1}{7} + 15t,\ 7t\right)$.

**(c)** For $a = 3$, $P: 2x - y + 3z = 5$.

$d = \frac{|2(1) - 0 + 3(2) - 5|}{\sqrt{2^2 + (-1)^2 + 3^2}} = \frac{|2 + 6 - 5|}{\sqrt{14}} = \frac{3}{\sqrt{14}}$.

**(d)** The foot is: $H = M - \frac{n \cdot M + d'}{\|n\|^2} n$.

Plane: $2x - y + 3z - 5 = 0$, so $d' = -5$, $n = (2,-1,3)$, $\| n \|^2 = 14$.

$n \cdot M = 2(1) - 0 + 3(2) = 8$.

$n \cdot M + d' = 8 - 5 = 3$.

$$\begin{aligned} H = (1,0,2) - \frac{3}{14}(2,-1,3) \\ &= \left(1 - \frac{6}{14},\ 0 + \frac{3}{14},\ 2 - \frac{9}{14}\right) \\ &= \left(\frac{8}{14},\ \frac{3}{14},\ \frac{19}{14}\right) \\ &= \left(\frac{4}{7},\ \frac{3}{14},\ \frac{19}{14}\right). \end{aligned}$$

$$\boxed{\begin{gathered} \text{(a) } a = 2,\ b = -6 \text{(b) } \left(\frac{17}{7} - 3t,\ -\frac{1}{7} + 15t,\ 7t\right) \\ \text{(c) } \frac{3}{\sqrt{14}} \text{(d) } \left(\frac{4}{7}, \frac{3}{14}, \frac{19}{14}\right) \end{gathered}}$$

## Problem 42 :

**Answer:** The angle $\phi$ between a line (direction $u$) and a plane (normal $n$) satisfies:

$$\sin\phi = \frac{| u \cdot n |}{\| u \| \| n \|}.$$

**(a)** $u = (1,2,-1)$, $n = (1,2,1)$.

$u \cdot n = 1 + 4 - 1 = 4$.

$\| u \| = \sqrt{1+4+1} = \sqrt{6}$, $\| n \| = \sqrt{1+4+1} = \sqrt{6}$.

$\sin\phi = \frac{4}{\sqrt{6} \cdot \sqrt{6}} = \frac{4}{6} = \frac{2}{3}$.

$\phi = \arcsin\left(\frac{2}{3}\right) \approx 41.8°$.

**(b)** Line direction: $u = (1,1,1)$. Plane $z = 0$ has normal $n = (0,0,1)$.

$\sin\phi = \frac{|1|}{\sqrt{3} \cdot 1} = \frac{1}{\sqrt{3}}$.

$\phi = \arcsin\left(\frac{1}{\sqrt{3}}\right) \approx 35.3°$.

**(c)** A line is perpendicular to a plane when its direction vector is parallel to the plane's normal vector: $u \parallel n$.

$$\boxed{\text{(a) } 41.8° \text{(b) } 35.3° \text{(c) } u \parallel n}$$

## Problem 43 :

**Answer:**

**(a)** Let $A(a_1,a_2,a_3)$ and $B(b_1,b_2,b_3)$. For a point $M(x,y,z)$, the condition $\| MA \|=\| MB \|$ is:

$$(x-a_1)^2+(y-a_2)^2+(z-a_3)^2 = (x-b_1)^2+(y-b_2)^2+(z-b_3)^2.$$

**(b)** Expand and cancel $x^2, y^2, z^2$:

$$-2a_1x+a_1^2-2a_2y+a_2^2-2a_3z+a_3^2 = -2b_1x+b_1^2-2b_2y+b_2^2-2b_3z+b_3^2.$$

$$2(b_1-a_1)x+2(b_2-a_2)y+2(b_3-a_3)z = b_1^2+b_2^2+b_3^2-a_1^2-a_2^2-a_3^2.$$

**(c)** This is a linear equation in $x,y,z$: $Ax+By+Cz=D$, which is the equation of a plane.

**(d)** The normal vector is $(b_1-a_1,\ b_2-a_2,\ b_3-a_3)=\overrightarrow{AB}$.

$$\boxed{\text{The locus is the perpendicular bisecting plane with normal } \overrightarrow{AB}}$$

## Problem 44 :

**Answer:**

**(a)** Let the centre of the sphere be at $O(0,0,0)$, apex at $(0,0,R)$.

The base is at height $z_0$ below the apex: $h=R+z_0$.

From the sphere equation $r^2+z_0^2=R^2$, so $r^2=R^2-z_0^2$.

**(b)** Substitute $z_0=h-R$:

$r^2+(h-R)^2=R^2-z_0^2+z_0^2=R^2$. ✓

**(c)** Let $h$ be the variable. Then $r^2=R^2-(h-R)^2=2Rh-h^2$.

Volume:

$$V(h)=\frac{\pi}{3}(2Rh-h^2)h=\frac{\pi}{3}(2Rh^2-h^3).$$

Differentiate:

$$\frac{dV}{dh}=\frac{\pi}{3}(4Rh-3h^2)=\frac{\pi h}{3}(4R-3h).$$

Set $dV/dh=0$: $h=0$ (minimum) or $h=\frac{4R}{3}$.

Thus the maximum volume occurs at $h=\frac{4R}{3}$.

**(d)** At $h=\frac{4R}{3}$:

$$r^2=2R\cdot\frac{4R}{3}-\left(\frac{4R}{3}\right)^2=\frac{8R^2}{3}-\frac{16R^2}{9}=\frac{8R^2}{9}.$$

$$V_{\max}=\frac{\pi}{3}\cdot\frac{8R^2}{9}\cdot\frac{4R}{3}=\frac{32\pi R^3}{81}.$$

Sphere volume: $V_{\text{sphere}}=\frac{4\pi R^3}{3}$.

Fraction:

$$\frac{V_{\max}}{V_{\text{sphere}}}=\frac{32\pi R^3}{81}\cdot\frac{3}{4\pi R^3}=\frac{96}{324}=\frac{8}{27}.$$

$$\boxed{\text{(c) } h=\frac{4R}{3}\text{(d) }\frac{8}{27}}$$

## Problem 45 :

**Answer:**

**(a)** $z=\frac{r^2}{4p}=\frac{0.5^2}{4(0.5)}=\frac{0.25}{2}=0.125$ m.

**(b)** Same as (a): the depth is $0.125$ m.

**(c)** $0.15=\frac{0.6^2}{4p}=\frac{0.36}{4p}\Rightarrow 4p=\frac{0.36}{0.15}=2.4\Rightarrow p=0.6$ m.

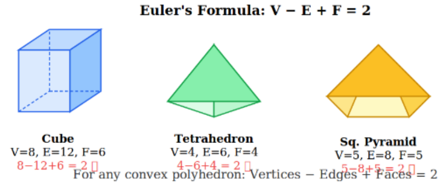
**(d)** Area of a dish is proportional to $r^2$:

Area ratio $=\left(\frac{1.2}{0.6}\right)^2=4$.

The larger dish has $4\times$ the collecting area and thus $4\times$ the signal power.

$$\boxed{\text{(a) }0.125\text{ m(b) }0.125\text{ m(c) }0.6\text{ m(d) }4\times}$$

## Problem 46 :



**Answer:**

**(a)** Cube: $V=8$, $E=12$, $F=6$. $8-12+6=2$ ✓.

**(b)** Tetrahedron: $V=4$, $E=6$, $F=4$. $4-6+4=2$ ✓.

**(c)** Square pyramid: $V=5$, $E=8$, $F=5$. $5-8+5=2$ ✓.

**(d)** $V-E+F=2\Rightarrow V-30+12=2\Rightarrow V=20$ vertices.

$$\boxed{\text{(a) 2(b) 2(c) 2(d) 20 vertices}}$$

## Problem 47 :

**Answer:**

**(a)** $3^2+4^2=9+16=25$ ✓. Yes, for any $z$.

**(b)** Parameterise the circle with $x=5\cos\theta$, $y=5\sin\theta$.

Then $z=x+2=5\cos\theta+2$.

Parametric form: $(5\cos\theta,\ 5\sin\theta,\ 5\cos\theta+2)$.

**(c)** The projection onto the $xy$-plane is the full circle $x^2+y^2=25$.

**(d)** A helix on the cylinder has parametric form:

$x=5\cos\theta,\ y=5\sin\theta,\ z=k\theta,$

where $k$ determines the pitch.

For example, with $k=2$: $x=5\cos\theta,\ y=5\sin\theta,\ z=2\theta$.

$$\boxed{\begin{array}{c}\text{(a) Yes(b) }(5\cos\theta,5\sin\theta,5\cos\theta+2)\\ \text{(c) Circle }x^2+y^2=25\text{(d) }(5\cos\theta,5\sin\theta,k\theta)\end{array}}$$

## Problem 48 :

**Answer:**

**(a)** $xyz=V_0$.

**(b)** $S=xy+2xz+2yz$ (base + four sides, but no top).

**(c)** With $x=y$: $z=\frac{V_0}{x^2}$.

$S(x)=x^2+4xz=x^2+\frac{4V_0}{x}$.

**(d)** Differentiate:

$$\frac{dS}{dx}=2x-\frac{4V_0}{x^2}=0\Rightarrow 2x^3=4V_0\Rightarrow x^3=2V_0.$$

$$x_{\text{opt}}=(2V_0)^{1/3}.$$

$$z=\frac{V_0}{x^2}=\frac{V_0}{(2V_0)^{2/3}}=\left(\frac{V_0}{4}\right)^{1/3}=\frac{x}{2}.$$

Optimal box: square base $x\times x$, height $z=\frac{x}{2}$.

$$\boxed{\begin{array}{c}\text{(a) } xyz = V_0 \text{(b) } S = xy + 2xz + 2yz \\ \text{(c) } S(x) = x^2 + \dfrac{4V_0}{x} \text{(d) } x = (2V_0)^{1/3},\ z = \dfrac{x}{2}\end{array}}$$

## Problem 49 :

**Answer:** The distance between skew lines is:

$$d = \frac{|\ \overrightarrow{AB} \cdot (u \times v)\ |}{\|\ u \times v\ \|}.$$

Compute $u \times v$:

$$\begin{aligned} u \times v &= (0 \cdot 1 - 1 \cdot 1,\ 1 \cdot 0 - 1 \cdot 1,\ 1 \cdot 1 - 0 \cdot 0) \\ &= (-1,-1,1). \\ \|\ u \times v\ \| &= \sqrt{1+1+1} = \sqrt{3}. \\ \overrightarrow{AB} &= B - A = (-1,1,0). \\ \overrightarrow{AB} \cdot (u \times v) &= (-1)(-1) + (1)(-1) + (0)(1) \\ &= 1 - 1 + 0 = 0. \end{aligned}$$

Since the dot product is $0$, the distance is $0$. This means the lines are **coplanar**. They must either intersect or be parallel. Since they are not parallel (vectors not proportional), they intersect. Solve $A + tu = B + sv$:

$$\begin{aligned} 1 + t = 0 &\Rightarrow t = -1, \\ 0 = 1 + s &\Rightarrow s = -1, \\ z\colon t = s &\Rightarrow -1 = -1\ \checkmark. \end{aligned}$$

Thus the lines intersect at $(0,0,-1)$.

$$\boxed{d = 0;\ \text{they intersect at } (0,0,-1)}$$

## Problem 50 :

**Answer: (a)** $\overrightarrow{AB} = (-2,2,1)$, $\overrightarrow{AC} = (-1,-1,3)$.

Cross product:

$$\begin{aligned} &AB \times AC \\ &= (2 \cdot 3 - 1 \cdot (-1),\quad 1 \cdot (-1) - (-2) \cdot 3,\quad (-2) \cdot (-1) - 2 \cdot (-1)) \\ &= (6 + 1,\ -1 + 6,\ 2 + 2) = (7,5,4). \end{aligned}$$

**(b)** Plane through $A(2,1,0)$ with normal $(7,5,4)$:
$7(x-2) + 5(y-1) + 4(z-0) = 0 \Rightarrow 7x + 5y + 4z = 19$.

**(c)** Centre $M$ = midpoint of $BC$ = $\left(\frac{0+1}{2}, \frac{3+0}{2}, \frac{1+3}{2}\right) = \left(\frac{1}{2}, \frac{3}{2}, 2\right)$.

Radius $r = \frac{\|BC\|}{2} = \frac{\sqrt{1^2+(-3)^2+2^2}}{2} = \frac{\sqrt{14}}{2}$.

Sphere: $\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{3}{2}\right)^2 + (z-2)^2 = \frac{14}{4}$.

**(d)** Compute $\|\ MA\ \|^2$:

$$A - M = \left(2 - \frac{1}{2},\ 1 - \frac{3}{2},\ 0 - 2\right) = \left(\frac{3}{2},\ -\frac{1}{2},\ -2\right).$$

$$\begin{aligned} \|\ MA\ \|^2 &= \left(\frac{3}{2}\right)^2 + \left(\ -\frac{1}{2}\right)^2 + (-2)^2 = \frac{9}{4} + \frac{1}{4} + 4 \\ &= \frac{10}{4} + 4 = \frac{5}{2} + 4 = \frac{13}{2} = 6.5. \end{aligned}$$

But $r^2 = \frac{14}{4} = 3.5$. Since $\|\ MA\ \|^2 > r^2$, $A$ is **outside** the sphere.

**(e)** Distance from $D(0,0,0)$ to plane $7x + 5y + 4z = 19$:

$$\begin{aligned} d &= \frac{|\ 7(0) + 5(0) + 4(0) - 19\ |}{\sqrt{7^2 + 5^2 + 4^2}} = \frac{19}{\sqrt{49 + 25 + 16}} \\ &= \frac{19}{\sqrt{90}} = \frac{19}{3\sqrt{10}} \approx 2.01. \end{aligned}$$

$$\boxed{\begin{array}{c}\text{(a) } (7,5,4) \text{(b) } 7x + 5y + 4z = 19 \\ \text{(c) } \left(x - \dfrac{1}{2}\right)^2 + \left(y - \dfrac{3}{2}\right)^2 + (z-2)^2 = \dfrac{14}{4} \\ \text{(d) Outside(e) } \dfrac{19}{3\sqrt{10}}\end{array}}$$

CHAPTER 10 — STATISTICS & PROBABILITY

# SOLUTIONS OF PROBLEMS

## Problems 1–15— Instant Recall

### Problem 1 :

**Answer:**

**(a)** Sum of scores: $5+8+12+8+15+10+12+8+14+8=100$.

Mean: $\bar{x}=\frac{100}{10}=10$.

**(b)** Ordered data: $5,\ 8,\ 8,\ 8,\ 8,\ 10,\ 12,\ 12,\ 14,\ 15$.

There are 10 values (even), so median is the average of the 5th and 6th values: $\frac{8+10}{2}=9$.

**(c)** Mode: the most frequent value is 8 (appears 4 times).

**(d)** Range: $15-5=10$.

$$\boxed{\text{(a) } 10 \text{(b) } 9 \text{(c) } 8 \text{(d) } 10}$$

### Problem 2 :

**Answer:**

**(a)** $2+5+4+3+1=15$ ✓

**(b)** Mean $=\frac{10\cdot2+12\cdot5+14\cdot4+16\cdot3+18\cdot1}{15}=\frac{20+60+56+48+18}{15}=\frac{202}{15}\approx13.47$.

$$\boxed{\text{(a) } 15 \text{(b) } \frac{202}{15}\approx13.47}$$

### Problem 3 :

**Answer:**

Total balls: $4+6+2=12$.

**(a)** P(red) $=\frac{4}{12}=\frac{1}{3}$.

**(b)** P(not blue) $=1-\frac{6}{12}=1-\frac{1}{2}=\frac{1}{2}$.

**(c)** P(red or green) $=\frac{4+2}{12}=\frac{6}{12}=\frac{1}{2}$.

**(d)** P(yellow) $=0$ (no yellow balls).

$$\boxed{\text{(a) } \frac{1}{3}\text{(b) } \frac{1}{2}\text{(c) } \frac{1}{2}\text{(d) } 0}$$

### Problem 4 :

**Answer:**

**(a)** $P(A')=1-0.35=0.65$

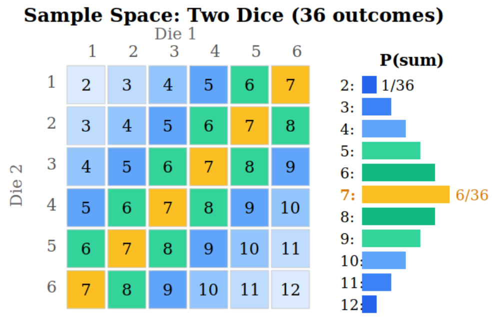
**(b)** $P(B)=1-0.72=0.28$

**(c)** P(no rain) $=1-0.6=0.4$

**(d)** $P(A')=1-p$

$$\boxed{\text{(a) } 0.65\text{(b) } 0.28\text{(c) } 0.4\text{(d) } 1-p}$$

### Problem 5 :



| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 6 | 7 | 8 | 9 | 10 | 11 | 12 |

Sum=7 is the most likely outcome: P(7)=6/36=1/6

**Answer:**

**(a)** Pairs summing to 7: (1,6),(2,5),(3,4),(4,3),(5,2),(6,1) → 6 outcomes.

P(sum = 7) $=\frac{6}{36}=\frac{1}{6}$.

**(b)** Only (6,6) sums to 12 → 1 outcome. P(sum = 12) $=\frac{1}{36}$.

**(c)** Sum ≤ 4: (1,1),(1,2),(2,1) → 3 outcomes (sum 2,3,3).

Also (1,3),(3,1) sum 4, and (2,2) sum 4. Let's list properly:

Sum 2: (1,1) → 1

Sum 3: (1,2),(2,1) → 2

Sum 4: (1,3),(2,2),(3,1) → 3

Total = $1+2+3=6$.

P(sum ≤ 4) $=\frac{6}{36}=\frac{1}{6}$.

**(d)** Both even: outcomes where both dice show 2,4,6 → $3\times3=9$ pairs.

P(both even) $=\frac{9}{36}=\frac{1}{4}$.

$$\boxed{\text{(a) } \frac{1}{6}\text{(b) } \frac{1}{36}\text{(c) } \frac{1}{6}\text{(d) } \frac{1}{4}}$$

### Problem 6 :

**Answer:**

**(a)** $P(A\cup B)=P(A)+P(B)-P(A\cap B)=0.4+0.5-0.2=0.7$.

**(b)** $P(A'\cap B')=P((A\cup B)')=1-P(A\cup B)=1-0.7=0.3$.

**(c)** $P(A\cap B)=0.2\neq0$, so they are **not** mutually exclusive.

**(d)** $P(A)\cdot P(B)=0.4\times0.5=0.2=P(A\cap B)$.

Therefore $A$ and $B$ **are independent**.

$$\boxed{\text{(a) } 0.7\text{(b) } 0.3\text{(c) No(d) Yes}}$$

### Problem 7 :

**Answer:**

**(a)** Sum of probabilities: $0.3+0.4+0.3=1$ ✓

**(b)** $E(X)=1\cdot0.3+2\cdot0.4+3\cdot0.3=0.3+0.8+0.9=2$.

**(c)** $E(X^2)=1^2\cdot0.3+2^2\cdot0.4+3^2\cdot0.3=0.3+1.6+2.7=4.6$.

**(d)** $\text{Var}(X)=E(X^2)-[E(X)]^2=4.6-2^2=4.6-4=0.6$.

$$\boxed{\text{(a) Valid(b) } 2\text{(c) } 4.6\text{(d) } 0.6}$$

## Problem 8 :

**Answer:**

**(a)** Sum: $2+4+4+4+5+5+7+9=40$.
$\bar{x}=\frac{40}{8}=5$.

**(b)** Deviations from mean: $-3,\ -1,\ -1,\ -1,\ 0,\ 0,\ 2,\ 4$.
Squared deviations: $9,\ 1,\ 1,\ 1,\ 0,\ 0,\ 4,\ 16$.
Sum of squares: $9+1+1+1+0+0+4+16=32$.
Variance: $\mathrm{Var}=\frac{32}{8}=4$.

**(c)** Standard deviation: $\sigma=\sqrt{4}=2$.

$$\boxed{\text{(a) }5\text{(b) }4\text{(c) }2}$$

## Problem 9 :

**Answer:**

**(a)** Relative frequencies:
$5/40=0.125, 8/40=0.2, 12/40=0.3, 10/40=0.25, 5/40=0.125$.

**(b)** Cumulative frequencies:
Value 1: 5, Value 2: 13, Value 3: 25, Value 4: 35, Value 5: 40.

**(c)** Frequency of scores ≤ 3 is $5+8+12=25$.
Percentage: $\frac{25}{40}\times 100=62.5\%$.

$$\boxed{\text{(a) }0.125,0.2,0.3,0.25,0.125\text{(b) }5,13,25,35,40\text{(c) }62.5\%}$$

## Problem 10 :

**Answer:**

**(a)** Cumulative frequencies:
Score 10: 3
Score 12: $3+8=11$
Score 14: $11+10=21$
Score 16: $21+6=27$
Score 18: $27+3=30$

**(b)** There are 30 values (even), so the median is the average of the 15th and 16th values.
Cumulative frequency reaches 21 at score 14, so both the 15th and 16th values are 14.
Median = 14.

$$\boxed{\text{(a) }3,11,21,27,30\text{(b) }14}$$

## Problem 11 :

**Answer:**

**(a)** P(heart) $=\frac{13}{52}=\frac{1}{4}$.

**(b)** P(king) $=\frac{4}{52}=\frac{1}{13}$.

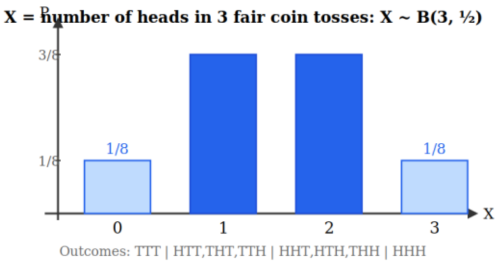
**(c)** P(heart or king) $=\frac{13}{52}+\frac{4}{52}-\frac{1}{52}=\frac{16}{52}=\frac{4}{13}$ (the king of hearts counted twice).

**(d)** P(heart and king) $=\frac{1}{52}$.

**(e)** P(neither heart nor king) $=1-\frac{4}{13}=\frac{9}{13}$.

$$\boxed{\text{(a) }\frac{1}{4}\text{(b) }\frac{1}{13}\text{(c) }\frac{4}{13}\text{(d) }\frac{1}{52}\text{(e) }\frac{9}{13}}$$

## Problem 12 :



**Answer:**

**(a)** Sample space:
$\{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$.

**(b)** Probabilities:
$P(X=0)=1/8$ (TTT)
$P(X=1)=3/8$ (HTT, THT, TTH)
$P(X=2)=3/8$ (HHT, HTH, THH)
$P(X=3)=1/8$ (HHH)

**(c)** $P(X\geq 2)=P(X=2)+P(X=3)=\frac{3}{8}+\frac{1}{8}=\frac{4}{8}=\frac{1}{2}$.

$$\boxed{\text{(a) }\{\ldots\}\text{(b) }1/8,3/8,3/8,1/8\text{(c) }\frac{1}{2}}$$

## Problem 13 :

**Answer:**

**(a)** Results 5 or 6 (2 outcomes) → win 4 DA: $P(X=4)=\frac{2}{6}=\frac{1}{3}$.
Results 1–4 (4 outcomes) → lose 2 DA: $P(X=-2)=\frac{4}{6}=\frac{2}{3}$.

**(b)** $E(X)=4\cdot\frac{1}{3}+(-2)\cdot\frac{2}{3}=\frac{4}{3}-\frac{4}{3}=0$.

**(c)** Since $E(X)=0$, the game is **fair**.

$$\boxed{\text{(a) }P(X=4)=\frac{1}{3},\ P(X=-2)=\frac{2}{3}\text{(b) }0\text{(c) Fair}}$$

## Problem 14 :

**Answer:**

**(a)** Since $A$ and $B$ are mutually exclusive, $P(A\cup B)=P(A)+P(B)=0.3+0.5=0.8$.

**(b)** $P(A'\cap B')=1-P(A\cup B)=1-0.8=0.2$.

**(c)** Mutually exclusive means $P(A\cap B)=0$.
Independence would require $P(A\cap B)=P(A)P(B)=0.3\times 0.5=0.15\neq 0$.
Thus mutually exclusive events with positive probabilities **cannot** be independent.

$$\boxed{\text{(a) }0.8\text{(b) }0.2\text{(c) No}}$$

.

## Problem 15 :

**Answer:** **(a)** Experimental P(heads) $=\frac{115}{200}=0.575$.

**(b)** Theoretical P(heads) $=0.5$.

**(c)** Expected heads in 200 tosses: $200\times 0.5=100$.
The difference is $115-100=15$ heads.
Standard deviation for 200 tosses: $\sigma=\sqrt{200\cdot 0.5\cdot 0.5}=\sqrt{50}\approx 7.07$.
The observed 115 is $(115-100)/7.07\approx 2.12$ standard deviations above the mean.
This is **unusual but not impossible** (about 1.7% chance if truly fair, assuming normal approximation).

**(d)** The law of large numbers states that as $n \to \infty$, the experimental relative frequency $\frac{\text{heads}}{n}$ converges to the theoretical probability $0.5$.

$$\boxed{\begin{gathered}\text{(a) } 0.575 \text{(b) } 0.5 \text{(c) Unusual but possible; expected } 100 \\ \text{(d) } \frac{\text{heads}}{n} \to 0.5\end{gathered}}$$

.

## Problems 16–38 — Medium

### Problem 16 :

**Answer:**

**(a)** Midpoints: $9, 11, 13, 15, 17$. Total students $N = 6 + 10 + 14 + 8 + 2 = 40$.

Mean:

$$\bar{x} = \frac{6 \cdot 9 + 10 \cdot 11 + 14 \cdot 13 + 8 \cdot 15 + 2 \cdot 17}{40} = \frac{54 + 110 + 182 + 120 + 34}{40} = \frac{500}{40} = 12.5.$$

**(b)** Variance:

$$= \frac{6(-3.5)^2 + 10(-1.5)^2 + 14(0.5)^2 + 8(2.5)^2 + 2(4.5)^2}{40} = \frac{6(12.25) + 10(2.25) + 14(0.25) + 8(6.25) + 2(20.25)}{40} = \frac{73.5 + 22.5 + 3.5 + 50 + 40.5}{40} = \frac{190}{40} = 4.75.$$

**(c)** Standard deviation: $\sigma = \sqrt{4.75} \approx 2.18$.

$$\boxed{\text{(a) } 12.5 \text{(b) } 4.75 \text{(c) } 2.18}$$

### Problem 17 :

**Answer:**

**(a)** $|F \cup A| = 18 + 15 - 7 = 26$.

**(b)** $P(F \cup A) = \frac{26}{30} = \frac{13}{15}$.

**(c)** Neither: $30 - 26 = 4$.

**(d)** French only: $18 - 7 = 11$.

$P(F \text{ only}) = \frac{11}{30}$.

$$\boxed{\text{(a) } 26 \text{(b) } \frac{13}{15} \text{(c) } 4 \text{(d) } \frac{11}{30}}$$

### Problem 18 :

**Answer:**

**(a)** $P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{0.24}{0.4} = 0.6$.

**(b)** $P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{0.24}{0.6} = 0.4$.

**(c)** Since $P(A \mid B) = 0.6 = P(A)$, the events are **independent**. (Also $P(A \cap B) = 0.24 = 0.6 \cdot 0.4$.)

**(d)** $P(A \cap B') = P(A) - P(A \cap B) = 0.6 - 0.24 = 0.36$.

$P(B') = 1 - 0.4 = 0.6$.

$P(A \mid B') = \frac{0.36}{0.6} = 0.6$.

$$\boxed{\text{(a) } 0.6 \text{(b) } 0.4 \text{(c) Yes(d) } 0.6}$$

### Problem 19 :

**Answer:**

**(a)** $P(X = 50000) = 0.02, P(X = 20000) = 0.05, P(X = 0) = 0.93$.

**(b)** $E(X) = 50000 \cdot 0.02 + 20000 \cdot 0.05 + 0 \cdot 0.93 = 1000 + 1000 + 0 = 2000$ DA.

**(c)** Expected profit per policy = Premium − $E(X) = 2500 - 2000 = 500$ DA.

$$\boxed{\begin{gathered}\text{(a) } P(50000) = 0.02,\ P(20000) = 0.05, \\ P(0) = 0.93 \text{(b) } 2000 \text{ DA (c) } 500 \text{ DA}\end{gathered}}$$

### Problem 20 :

**Answer:**

**(a)** $P(D) = P(D \mid M_1)P(M_1) + P(D \mid M_2)P(M_2) = 0.02 \cdot 0.6 + 0.05 \cdot 0.4 = 0.012 + 0.020 = 0.032$.

**(b)** $P(M_2 \mid D) = \frac{P(D|M_2)P(M_2)}{P(D)} = \frac{0.05 \cdot 0.4}{0.032} = \frac{0.020}{0.032} = 0.625$.

$$\boxed{\text{(a) } 0.032 \text{(b) } 0.625}$$

### Problem 21 :

**Answer:** **(a)**

$$E(aX + b) = \sum(ax + b)P(X = x) = a\sum xP(X = x) + b\sum P(X = x) = aE(X) + b.$$

**(b)** Let $\mu = E(X)$.

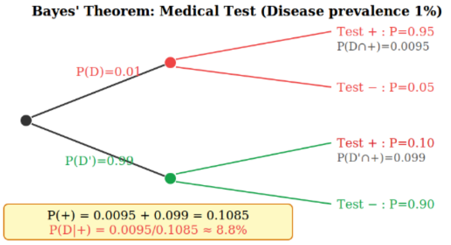
$$\text{Var}(aX + b) = E[(aX + b - a\mu - b)^2] = E[a^2(X - \mu)^2] = a^2\text{Var}(X).$$

**(c)** $E(5X - 1) = 5 \cdot 3 - 1 = 14$.

$\sigma(5X - 1) = |5| \cdot \sigma(X) = 5 \cdot 2 = 10$.

$$\boxed{\begin{gathered}\text{(a) } E(aX + b) = aE(X) + b \\ \text{(b) } \text{Var}(aX + b) = a^2\text{Var}(X) \\ \text{(c) } 14,\ 10\end{gathered}}$$

### Problem 22 :


Bayes' Theorem: Medical Test (Disease prevalence 1%)
P(D)=0.01
P(D')=0.99
Test + : P=0.95
P(D∩+)=0.0095
Test − : P=0.05
Test + : P=0.10
P(D'∩+)=0.099
Test − : P=0.90
P(+) = 0.0095 + 0.099 = 0.1085
P(D|+) = 0.0095/0.1085 ≈ 8.8%

**Answer:**

**(a)** $P(+) = P(+| D)P(D) + P(+| D')P(D') = 0.95 \cdot 0.01 + 0.10 \cdot 0.99 = 0.0095 + 0.099 = 0.1085$.

**(b)** $P(D \mid +) = \frac{P(+|D)P(D)}{P(+)} = \frac{0.0095}{0.1085} \approx 0.0876$ (about 8.76%).

$$\boxed{\text{(a) } 0.1085 \text{(b) } 0.0876 \approx 8.76\%}$$

### Problem 23 :

**Answer:** **(a)** The class with the highest frequency is $[20,30)$ with 18 students.

**(b)** Midpoints: $5, 15, 25, 35, 45$.

$$\text{Mean} = \frac{5\cdot 5 + 12\cdot 15 + 18\cdot 25 + 10\cdot 35 + 5\cdot 45}{50}$$
$$= \frac{25 + 180 + 450 + 350 + 225}{50}$$
$$= \frac{1230}{50} = 24.6.$$

**(c)** Cumulative frequencies: $5,\ 17,\ 35,\ 45,\ 50$.
The 25th value is in $[20{,}30)$.
Estimated median: $20 + \frac{25-17}{18}\cdot 10 = 20 + \frac{8}{18}\cdot 10 \approx 24.4$.
**(d)** Frequency polygon connects the midpoints: $(5{,}5), (15{,}12), (25{,}18), (35{,}10), (45{,}5)$.

$$\boxed{\begin{array}{c}\text{(a) } [20{,}30)\text{(b) } 24.6\text{(c) } [20{,}30),\ \approx 24.4\\ \text{(d) connect midpoints}\end{array}}$$

## Problem 24 :

**Answer:**
**(a)** $P(\text{all}) = 0.8\cdot 0.7\cdot 0.6 = 0.336$.
**(b)** $P(\text{none}) = 0.2\cdot 0.3\cdot 0.4 = 0.024$.
**(c)** Exactly two pass:
$0.8\cdot 0.7\cdot 0.4 + 0.8\cdot 0.3\cdot 0.6 + 0.2\cdot 0.7\cdot 0.6 = 0.224 + 0.144 + 0.084 = 0.452$.
**(d)** $P(\text{at least one}) = 1 - P(\text{none}) = 1 - 0.024 = 0.976$.

$$\boxed{\text{(a) } 0.336\text{(b) } 0.024\text{(c) } 0.452\text{(d) } 0.976}$$

## Problem 25 :

**Answer:**
**(a)** $E(X) = -1\cdot\frac{1}{4} + 0\cdot\frac{1}{2} + 1\cdot\frac{1}{4} = -\frac{1}{4} + 0 + \frac{1}{4} = 0$.
**(b)** $E(X^2) = 1\cdot\frac{1}{4} + 0\cdot\frac{1}{2} + 1\cdot\frac{1}{4} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.
**(c)** $\text{Var}(X) = E(X^2) - [E(X)]^2 = \frac{1}{2} - 0 = \frac{1}{2}$.
**(d)** $E(3X^2 + 2X - 1) = 3E(X^2) + 2E(X) - 1 = 3\cdot\frac{1}{2} + 2\cdot 0 - 1 = \frac{3}{2} - 1 = \frac{1}{2}$.

$$\boxed{\text{(a) } 0\text{(b) } \frac{1}{2}\text{(c) } \frac{1}{2}\text{(d) } \frac{1}{2}}$$

## Problem 26 :

**Answer:**
**(a)** Midpoints: $152.5, 157.5, 162.5, 167.5, 172.5$.
Mean
$$= \frac{4(152.5) + 8(157.5) + 14(162.5) + 10(167.5) + 4(172.5)}{40}$$
$$= \frac{610 + 1260 + 2275 + 1675 + 690}{40} = \frac{6510}{40}$$
$= 162.75$ cm.

**(b)** Cumulative: $4,\ 12,\ 26,\ 36,\ 40$.
**(c)** 20th value is in $[160{,}165)$:
$160 + \frac{20-12}{14}\cdot 10 = 160 + \frac{8}{14}\cdot 10 \approx 162.86$ cm.
**(d)** Variance:
Var
$$= \frac{4(-10.25)^2 + 8(-5.25)^2 + 14(-0.25)^2 + 10(4.75)^2 + 4(9.75)^2}{40}$$
$$= \frac{420.25 + 220.5 + 0.875 + 225.25 + 380.25}{40}$$
$$= \frac{1247.125}{40} \approx 31.18.$$

$\sigma \approx \sqrt{31.18} \approx 5.58$.

$$\boxed{\begin{array}{c}\text{(a) } 162.75 \text{ cm(b) } 4{,}12{,}26{,}36{,}40\\ \text{(c) } 162.86 \text{ cm (d) } \approx 5.58 \text{ cm}\end{array}}$$

## Problem 27 :

**Answer:** **(a)** Area of square $= 16$, area of circle $= \pi(1)^2 = \pi$.
$P(\text{inside circle}) = \frac{\pi}{16}$.
**(b)** The inner region $> 1$ cm from all edges is a $2\times 2$ square of area $4$.
$P(\text{within 1 cm of edge}) = 1 - \frac{4}{16} = \frac{3}{4}$.
**(c)** The region $x > 3$ is a strip of width $1$ and height $4$, area $4$.
$P(x > 3) = \frac{4}{16} = \frac{1}{4}$.

$$\boxed{\text{(a) } \frac{\pi}{16}\text{(b) } \frac{3}{4}\text{(c) } \frac{1}{4}}$$

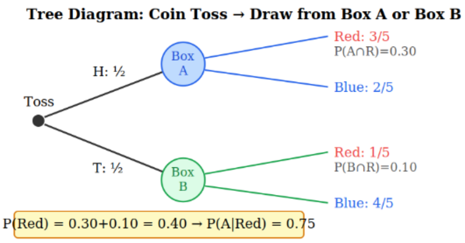
## Problem 28 :

**Answer:**
**(a)** Approximately 68% of scores fall within one standard deviation.
Number of students: $0.68 \times 35 \approx 24$ students in $[9.2, 15.6]$.
**(b)** Range: $[12.4 - 3.2,\ 12.4 + 3.2] = [9.2,\ 15.6]$.
**(c)** Two standard deviations: $2\sigma = 6.4$.
Threshold for unusual (beyond 2σ): $12.4 + 6.4 = 18.8$.

- Score 18: within 2σ, not unusual.
- Scores 19 and 20: exceed 18.8, so they are statistically unusual (about top 2.5% of a normal distribution).

$$\boxed{\begin{array}{c}\text{(a) } \approx 24 \text{ students(b) } [9.2{,}15.6]\\ \text{(c) 19 and 20 are unusual}\end{array}}$$

## Problem 29 :


Tree Diagram: Coin Toss → Draw from Box A or Box B
Toss
H: ½
T: ½
Box A
Box B
Red: 3/5
P(A∩R)=0.30
Blue: 2/5
Red: 1/5
P(B∩R)=0.10
Blue: 4/5
P(Red) = 0.30+0.10 = 0.40 → P(A|Red) = 0.75

**Answer:**
**(a)** $P(\text{red}) = P(H)\cdot P(R\mid A) + P(T)\cdot P(R\mid B) = 0.5\cdot\frac{3}{5} + 0.5\cdot\frac{1}{5} = 0.3 + 0.1 = 0.4$.
**(b)** $P(A\mid R) = \frac{P(R|A)P(A)}{P(R)} = \frac{0.5\cdot\frac{3}{5}}{0.4} = \frac{0.3}{0.4} = 0.75$.

$$\boxed{\text{(a) } 0.4\text{(b) } 0.75}$$

## Problem 30 :

**Answer:**
**(a)** $P(X = 0) = \left(\frac{5}{6}\right)^4 = \frac{625}{1296} \approx 0.482$.
**(b)** $P(X = 1) = \binom{4}{1}\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^3 = 4\cdot\frac{1}{6}\cdot\frac{125}{216} = \frac{500}{1296} \approx 0.386$.
**(c)** $P(X \geq 1) = 1 - P(X = 0) = 1 - \frac{625}{1296} = \frac{671}{1296} \approx 0.518$.
**(d)** $E(X) = np = 4\cdot\frac{1}{6} = \frac{2}{3}$.

$$\boxed{\text{(a) } 0.482 \text{(b) } 0.386 \text{(c) } 0.518 \text{(d) } \frac{2}{3}}$$

## Problem 31 :

**Answer:**

**(a)** Sum $= 15 + 18 + 20 + 22 + 25 + 25 + 28 + 30 + 200 = 383.$

Mean $= \frac{383}{9} \approx 42.6$ thousand DA.

**(b)** Ordered data: $15,18,20,22,25,25,28,30,200.$
Median (5th value) $= 25$ thousand DA.
**(c)** The median better represents the typical salary, as it is not distorted by the outlier 200.
**(d)** Without 200: sum $= 183$, mean $= \frac{183}{8} = 22.875$ thousand DA.
Median of the 8 values: average of 4th and 5th $= \frac{22+25}{2} = 23.5$ thousand DA.
The mean drops dramatically; the median changes only slightly.

$$\boxed{\text{(a) } 42.6 \text{(b) } 25 \text{(c) median(d) mean: } 22.9\text{, median: } 23.5}$$

## Problem 32 :

**Answer:**

**(a)** $E(X) = \frac{3+5+7+9+11}{5} = \frac{35}{5} = 7.$

**(b)** $E(X^2) = \frac{9+25+49+81+121}{5} = \frac{285}{5} = 57.$

**(c)** $\mathrm{Var}(X) = 57 - 7^2 = 57 - 49 = 8.$
$\sigma = \sqrt{8} = 2\sqrt{2} \approx 2.83.$

$$\boxed{\text{(a) } 7 \text{(b) } 57 \text{(c) } 8,\ 2\sqrt{2}}$$

## Problem 33 :

**Answer:**

**(a) With replacement:**

$P(RR) = \left(\frac{5}{8}\right)^2 = \frac{25}{64}.$

$P(\text{one of each}) = 2 \cdot \frac{5}{8} \cdot \frac{3}{8} = \frac{30}{64} = \frac{15}{32}.$

**(b) Without replacement:**

$P(RR) = \frac{5}{8} \cdot \frac{4}{7} = \frac{20}{56} = \frac{5}{14}.$

$P(\text{one of each}) = 2 \cdot \frac{5}{8} \cdot \frac{3}{7} = \frac{30}{56} = \frac{15}{28}.$

$$\boxed{\text{(a) } RR\text{:}\frac{25}{64}\text{, one:}\frac{15}{32} \text{(b) } RR\text{:}\frac{5}{14}\text{, one:}\frac{15}{28}}$$

## Problem 34 :

**Answer:**

**(a)** Subtract ticket cost:
$P(X = 9800) = \frac{1}{100}, P(X = 300) = \frac{5}{100}, P(X = -200) = \frac{94}{100}.$

**(b)** $E(X) = 9800 \cdot 0.01 + 300 \cdot 0.05 + (-200) \cdot 0.94 = 98 + 15 - 188 = -75$ DA.
**(c)** Expected loss of 75 DA per ticket: **not worth it** on average.

$$\boxed{\begin{array}{l}\text{(a) } P(9800) = 0.01,\ P(300) = 0.05, \\ P(-200) = 0.94 \text{(b) } -75 \text{ DA (c) No}\end{array}}$$

## Problem 35 :

**Answer:**

Let $\mu = E(X)$. Then

$$\begin{aligned}\mathrm{Var}(X) &= E[(X-\mu)^2] = E[X^2 - 2\mu X + \mu^2] \\ &= E(X^2) - 2\mu E(X) + \mu^2.\end{aligned}$$

Since $\mu = E(X)$, this becomes:

$$\begin{aligned}\mathrm{Var}(X) &= E(X^2) - 2\mu^2 + \mu^2 = E(X^2) - \mu^2 \\ &= E(X^2) - [E(X)]^2.\end{aligned}$$

$$\boxed{\mathrm{Var}(X) = E(X^2) - [E(X)]^2}$$

## Problem 36 :

**Answer:** **(a)** Two-way table:

| | Pass | Fail | Total |
|---|---|---|---|
| Male | 70 | 20 | 90 |
| Female | 80 | 30 | 110 |
| Total | 150 | 50 | 200 |

**(b)** $P(\text{pass}) = \frac{150}{200} = \frac{3}{4}.$

**(c)** $P(\text{pass} \mid \text{male}) = \frac{70}{90} = \frac{7}{9} \approx 0.78.$

**(d)** $P(\text{male} \mid \text{pass}) = \frac{70}{150} = \frac{7}{15} \approx 0.47.$

**(e)** $P(\text{pass}) \cdot P(\text{male}) = \frac{3}{4} \cdot \frac{90}{200} = \frac{3}{4} \cdot \frac{9}{20} = \frac{27}{80} = 0.3375.$

$P(\text{pass} \cap \text{male}) = \frac{70}{200} = 0.35 \neq 0.3375.$

**Not independent.**

$$\boxed{\text{(a) table above(b) } \frac{3}{4}\text{(c) } \frac{7}{9}\text{(d) } \frac{7}{15}\text{(e) No}}$$

## Problem 37 :

**Answer:**

**(a)** $P(+) = 0.99 \cdot 0.001 + 0.05 \cdot 0.999 = 0.00099 + 0.04995 = 0.05094.$

**(b)** $P(C \mid +) = \frac{0.00099}{0.05094} \approx 0.0194$ (about 1.94%).

**(c)** Even with a 99% accurate test, most positives are **false positives** when the disease is very rare.
This shows that mass screening of low-prevalence conditions produces mainly false alarms.

$$\boxed{\text{(a) } 0.05094 \text{(b) } 0.0194\ (1.94\%) \text{(c) mostly false positives}}$$

## Problem 38 :

**Answer:**

**(a)** $E(Y) = 2 \cdot 50 - 5 = 95.$
**(b)** $\sigma(Y) =\mid 2 \mid \cdot\ 10 = 20.$
**(c)** $\mathrm{Var}(Y) = 2^2 \cdot 10^2 = 4 \cdot 100 = 400.$
**(d)** $E(Z) = 50 + 8 = 58$; $\sigma(Z) = 10$ (adding a constant does not change the spread).

$$\boxed{\text{(a) } 95 \text{(b) } 20 \text{(c) } 400 \text{(d) } 58,\ 10}$$

# Problems 39–50 — Difficult & Real-World

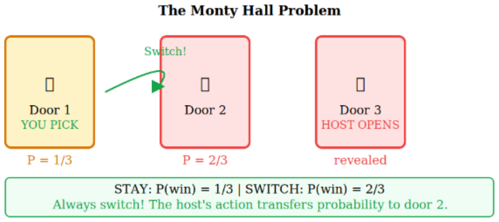
## Problem 39 :

**Answer:**

**(a)** $A$ and $A'$ are mutually exclusive, and their union is the entire sample space:
$A \cup A' = \Omega$.
**(b)** By the axiom for mutually exclusive events:
$P(A \cup A') = P(A) + P(A')$.
**(c)** Since $P(\Omega) = 1$ (the total probability axiom), we have:
$1 = P(A) + P(A')$.
**(d)** Subtract $P(A)$ from both sides:
$P(A') = 1 - P(A)$. ✓

$$\boxed{P(A') = 1 - P(A)}$$

## Problem 40 :


The Monty Hall Problem
Switch!
Door 1
YOU PICK
P = 1/3
Door 2
P = 2/3
Door 3
HOST OPENS
revealed
STAY: P(win) = 1/3 | SWITCH: P(win) = 2/3
Always switch! The host's action transfers probability to door 2.

**Answer:**

**(a)** Before any doors are opened, the probabilities are:
$P(\text{prize at 1}) = \frac{1}{3}, P(\text{prize at 2}) = \frac{1}{3}, P(\text{prize at 3}) = \frac{1}{3}$.
**(b)** Let $H$ be the event that the host opens door 3.

- If the prize is at door 1, the host randomly opens door 2 or 3: $P(H \mid \text{door 1}) = \frac{1}{2}$.
- If the prize is at door 2, the host must open door 3: $P(H \mid \text{door 2}) = 1$.
- If the prize is at door 3, the host cannot open door 3 (it has the prize): $P(H \mid \text{door 3}) = 0$.

Using the law of total probability:
$P(H) = \frac{1}{2} \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} = \frac{1}{6} + \frac{1}{3} = \frac{1}{2}$.
Now apply Bayes' theorem to find the probability that the prize is behind door 2 given that the host opened door 3:

$P(\text{door 2} \mid H) = \frac{P(H|\text{door 2})P(\text{door 2})}{P(H)} = \frac{1 \cdot \frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$.
**(c)** If you switch to door 2, you win with probability $\frac{2}{3}$.

If you stay with door 1, you win with probability $\frac{1}{3}$.
Therefore, **you should switch**.
**(d)** The host's action provides information: he deliberately avoids the door with the prize. The door he did not open (door 2) is more likely to contain the prize.

$$\boxed{\text{Switch; win probability } \frac{2}{3}}$$

## Problem 41 :

**Answer:**

**(a)** If $\text{Var}(X) = 0$, then by definition $\text{Var}(X) = E(X^2) - \mu^2 = 0$, so $E(X^2) = \mu^2$.
**(b)** The variance can also be written as:
$\text{Var}(X) = \sum_i (x_i - \mu)^2 P(X = x_i)$.
**(c)** Since each term $(x_i - \mu)^2 \geq 0$ and $P(X = x_i) \geq 0$, the sum is zero only if every term is zero.
**(d)** Thus for every outcome with positive probability, $(x_i - \mu)^2 = 0$, which means $x_i = \mu$.
Therefore, all possible values of $X$ equal the mean $\mu$. Conversely, if all values equal $\mu$, then clearly the variance is zero.

$$\boxed{\sigma(X) = 0 \iff \text{all values equal } \mu}$$

## Problem 42 :

**Answer:**

**(a)**

$$P(X = 3) = \binom{10}{3}(0.3)^3(0.7)^7$$
$$= 120 \cdot 0.027 \cdot 0.0824 \approx 0.2668.$$

**(b)**

$$P(X \geq 8) = P(X = 8) + P(X = 9) + P(X = 10)$$
$$= \binom{10}{8}(0.3)^8(0.7)^2 + \binom{10}{9}(0.3)^9(0.7)^1$$
$$+ \binom{10}{10}(0.3)^{10}(0.7)^0$$
$$\approx 0.0014 + 0.0001 + 0.000006 \approx 0.0015.$$

**(c)**
$E(X) = np = 10(0.3) = 3$.
$\sigma(X) = \sqrt{np(1-p)} = \sqrt{10(0.3)(0.7)} = \sqrt{2.1} \approx 1.45$.
**(d)** Start with the binomial expansion:

$$\sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} = (p+q)^n = 1.$$

Differentiate both sides with respect to $p$:

$$\sum_{k=0}^{n} \binom{n}{k} k p^{k-1} q^{n-k} = n(p+q)^{n-1}.$$

Multiply both sides by $p$ (and note $q = 1 - p$ so $p + q = 1$):

$$\sum_{k=0}^{n} k \binom{n}{k} p^k q^{n-k} = np(1)^{n-1} = np.$$

The left side is exactly $E(X)$. Hence $E(X) = np$. ✓

$$\boxed{\begin{array}{c}\text{(a) } 0.2668 \text{(b) } 0.0015 \text{(c) } E = 3,\ \sigma \approx 1.45 \\ \text{(d) } E(X) = np\end{array}}$$

## Problem 43 :

**Answer:** **(a)** For $k = 2$:
$P(|X - \mu| \geq 2\sigma) \leq \frac{1}{4}$, so
$P(|X - \mu| < 2\sigma) \geq 1 - \frac{1}{4} = \frac{3}{4} = 75\%$.
**(b)** Here $k = 3$ (since $3\sigma = 15$).
$P(|X - 50| \geq 15) \leq \frac{1}{9}$, so

$P(35 \le X \le 65) = P(|X - 50| < 15) \ge 1 - \frac{1}{9} = \frac{8}{9} \approx 88.9\%$.

**(c)** For a normal distribution, the empirical rule gives 99.7% within 3 standard deviations. Chebyshev gives only 88.9% — much weaker.

**(d)** Chebyshev's inequality works for **any** distribution (as long as the mean and variance are finite), while the empirical rule only applies to approximately normal distributions. The generality comes at the cost of a looser bound.

(a) 75%(b) $\ge 88.9\%$
(c) Chebyshev weaker but universal
(d) works for all distributions

## Problem 44 :

**Answer:**

**(a)** Sum: $0.5 + 0.3 + 0.15 + 0.05 = 1$. ✓

**(b)**
$E(X) = 0 \cdot 0.5 + 1 \cdot 0.3 + 2 \cdot 0.15 + 3 \cdot 0.05 = 0.3 + 0.3 + 0.15 = 0.75$.
$E(X^2) = 0 \cdot 0.5 + 1 \cdot 0.3 + 4 \cdot 0.15 + 9 \cdot 0.05 = 0.3 + 0.6 + 0.45 = 1.35$.
$\text{Var}(X) = 1.35 - (0.75)^2 = 1.35 - 0.5625 = 0.7875$.
$\sigma(X) = \sqrt{0.7875} \approx 0.887$.

**(c)** $P(X \le 1) = 0.5 + 0.3 = 0.8$.
$P(X \ge 2) = 0.15 + 0.05 = 0.2$.

**(d)** Since days are independent:
$P(0 \text{ defects on all 3 days}) = 0.5^3 = 0.125$.

**(e)** Expected total over 5 days: $5 \cdot E(X) = 5 \cdot 0.75 = 3.75$ defectives.

(a) valid(b) $E = 0.75$, $\text{Var} = 0.7875$, $\sigma \approx 0.887$
(c) $0.8$, $0.2$(d) $0.125$(e) $3.75$

## Problem 45 :

**Answer:**

**(a)**
There are 4 aces: $P(X = 10) = \frac{4}{52} = \frac{1}{13}$.

There are 12 face cards (J,Q,K): $P(X = 5) = \frac{12}{52} = \frac{3}{13}$.

Remaining $52 - 4 - 12 = 36$ cards: $P(X = -2) = \frac{36}{52} = \frac{9}{13}$.

**(b)**
$E(X) = 10 \cdot \frac{1}{13} + 5 \cdot \frac{3}{13} - 2 \cdot \frac{9}{13} = \frac{10+15-18}{13} = \frac{7}{13} \approx 0.54$ DA.

**(c)**
$E(X^2) = 100 \cdot \frac{1}{13} + 25 \cdot \frac{3}{13} + 4 \cdot \frac{9}{13} = \frac{100+75+36}{13} = \frac{211}{13}$.

$\text{Var}(X) = E(X^2) - [E(X)]^2 = \frac{211}{13} - \left(\frac{7}{13}\right)^2 = \frac{211}{13} - \frac{49}{169} = \frac{2743-49}{169} = \frac{2694}{169} \approx 15.94$.

**(d)** Expected total over 100 games: $100 \cdot E(X) = 100 \cdot \frac{7}{13} \approx 53.8$ DA.

(a) $P(10) = \frac{1}{13}$, $P(5) = \frac{3}{13}$, $P(-2) = \frac{9}{13}$
(b) $\frac{7}{13}$ DA(c) 15.94(d) 53.8 DA

## Problem 46 :

**Answer:**

**(a)**
$\bar{x} = \frac{2+3+4+5+6+8+10}{7} = \frac{38}{7} \approx 5.43$ hours.
$\bar{y} = \frac{40+50+55+65+70+80+90}{7} = \frac{450}{7} \approx 64.29$.

**(b)** Compute deviations and products:
$\sum(x_i - \bar{x})(y_i - \bar{y}) \approx 139.7$.
$\sum(x_i - \bar{x})^2 \approx 36.9$.
$\sum(y_i - \bar{y})^2 \approx 530$.

**(c)**

$$r = \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum(x_i - \bar{x})^2 \cdot \sum(y_i - \bar{y})^2}} = \frac{139.7}{\sqrt{36.9 \cdot 530}} = \frac{139.7}{\sqrt{19557}} \approx \frac{139.7}{139.8} \approx 0.999.$$

**(d)** $r \approx 1$ indicates a **near-perfect positive linear correlation** between study time and exam scores: as study hours increase, scores increase almost linearly.

(a) $\bar{x} \approx 5.43$, $\bar{y} \approx 64.29$(b) 139.7, 36.9, 530
(c) 0.999(d) near-perfect positive correlation

## Problem 47 :

**Answer:**

**(a)**
$E(X) = \int_0^{10} x \cdot \frac{1}{10}\, dx = \frac{1}{10}\left[\frac{x^2}{2}\right]_0^{10} = \frac{1}{10} \cdot \frac{100}{2} = 5$.

**(b)**
$E(X^2) = \int_0^{10} x^2 \cdot \frac{1}{10}\, dx = \frac{1}{10}\left[\frac{x^3}{3}\right]_0^{10} = \frac{1}{10} \cdot \frac{1000}{3} = \frac{100}{3}$.
$\text{Var}(X) = E(X^2) - [E(X)]^2 = \frac{100}{3} - 25 = \frac{100}{3} - \frac{75}{3} = \frac{25}{3} \approx 8.33$.

**(c)**
$P(2 \le X \le 7) = \frac{7-2}{10} = 0.5$.

**(d)**
$X^2 \le 25 \iff |X| \le 5 \iff 0 \le X \le 5$ (since $X \ge 0$).
$P(X \le 5) = \frac{5}{10} = 0.5$.

(a) 5(b) $\frac{25}{3}$(c) 0.5(d) 0.5

## Problem 48 :

**Answer:**

**(a)** Absolute deviation: $|0.53 - 0.5| = 0.03$.

**(b)** $\frac{1}{2\sqrt{100}} = \frac{1}{20} = 0.05$.

**(c)** $0.03 < 0.05$, so 0.53 is **within 1 standard deviation** of 0.5. This is a very typical result.

**(d)** Solve $\frac{1}{2\sqrt{n}} < 0.01$:
$2\sqrt{n} > 100 \Rightarrow \sqrt{n} > 50 \Rightarrow n > 2500$.
So for $n > 2500$, the standard deviation falls below 0.01.

(a) 0.03(b) 0.05(c) Yes(d) $n > 2500$

## Problem 49 :

**Answer: (a)**

$P(A \cap B') = P(A) - P(A \cap B) = P(A) - P(A)P(B) = P(A)(1 - P(B)) = P(A)P(B')$.

Hence $A$ and $B'$ are independent. ✓

**(b)**

$P(A' \cap B') = P((A \cup B)') = 1 - P(A \cup B)$
$= 1 - [P(A) + P(B) - P(A \cap B)]$
$= 1 - P(A) - P(B) + P(A)P(B)$
$= (1 - P(A))(1 - P(B)) = P(A')P(B')$.

Hence $A'$ and $B'$ are independent. ✓

**(c)** If $A$ and $B$ are mutually exclusive, then $P(A \cap B) = 0$.

For independence, we would need $P(A \cap B) = P(A)P(B)$.

Since $P(A) > 0$ and $P(B) > 0$, their product is positive, so it cannot equal 0.

Thus **mutually exclusive events with positive probabilities cannot be independent** (they are incompatible notions except for trivial cases where one event has probability 0).

$$\boxed{\begin{array}{c}\text{(a) } A \text{ and } B' \text{ independent} \\ \text{(b) } A' \text{ and } B' \text{ independent(c) No}\end{array}}$$

## Problem 50 :

**Answer:**

**(a)**

$P(M)P(P) = 0.6 \cdot 0.5 = 0.3$, but $P(M \cap P) = 0.35 \neq 0.3$.

Thus M and P are **not independent**.

**(b)**

$P(M \cup P) = 0.6 + 0.5 - 0.35 = 0.75$.
$P(\text{neither}) = 1 - 0.75 = 0.25$.

**(c)**

$P(\text{both} \mid \text{at least one}) = \frac{P(M \cap P)}{P(M \cup P)} = \frac{0.35}{0.75} = \frac{7}{15} \approx 0.467$.

**(d)**

New score: $Y = 5X$.
$E(Y) = 5E(X) = 5 \cdot 14 = 70$.
$\sigma(Y) = |5| \cdot \sigma(X) = 5 \cdot 2 = 10$.
So on a 0–100 scale, the mean is 70 and the standard deviation is 10

.

$$\boxed{\begin{array}{c}\text{(a) Not independent(b) } 0.75,\ 0.25\text{(c) } \dfrac{7}{15} \\ \text{(d) mean } 70,\ \sigma = 10\end{array}}$$

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